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STABILITY, DISEQUILIBRIUM AWARENESS, AND
THE PERCEPTION OF NEW OPPORTUNITIES¹

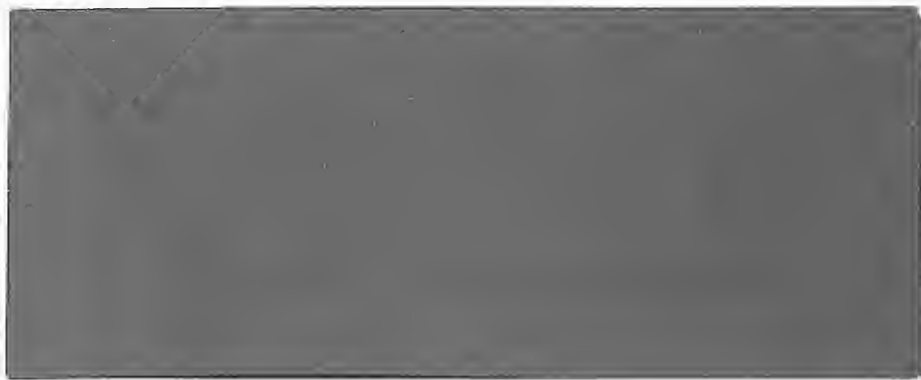
Franklin M. Fisher

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**massachusetts
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**50 memorial drive
cambridge, mass.02139**



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¹ Paper to be presented as the Econometric Society Presidential Address, 1979.

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PERCEPTION OF NEW OPPORTUNITIES¹

Franklin M. Fisher
Massachusetts Institute of Technology

1. Paper to be presented as the Econometric Society Presidential Address, 1979.

1. Introduction

One of the central unsolved problems of micro-economics is the construction of a satisfactory adjustment model which converges to equilibrium as a result of rational behavior on the part of the decision-making agents involved. This is not a simple matter, particularly when production and consumption are allowed to go on outside of equilibrium.²

2. Models with explicit individual price adjustment are examined in Fisher [10, 11, 12]. The introduction, first of firms and then of production and consumption is in Fisher [13, 14, 16]. See also [15]. Fisher [17] gives a relatively non-technical survey of the subject and discusses a number of the problems involved.

Principal among the difficulties has been the persistent assumption that agents expect current prices to persist despite continual evidence of price change. When coupled with the other usual assumption that agents also expect to be able costlessly to complete all their desired transactions, this means that agents stupidly believe that they are in equilibrium when it must be obvious to them that they are not.³

3. See Fisher [17] for a somewhat unsatisfactory attempt at having agents recognize that their transactions may not be completed but in which price expectations remain naïve.

The present paper attempts to rectify some of these failings. In it, agents are allowed to have more sensible price expectations. Those expectations can be formed in a variety of ways so long as they are revised in accordance with a particular assumption to be described below. The class of behavior permitted is wide enough to include perfect foresight as well as naïve expectations of the sort already described. While the permitted behavior is still quite restrictive, this seems to me to be a considerable gain over having to assume that agents believe current prices will last forever.

This paper also drops the assumption that agents believe they can costlessly complete their transactions. Instead, agents are assumed to believe either that they face quantity constraints on their transactions or, more generally, that there are utility or resource costs associated with transaction size. Agents optimize their actions taking such matters fully into account so that so-called "spillover" effects are included. Obviously, it matters a good deal how agents form their perceptions of transaction costs or transaction constraints, and a fairly restrictive assumption, similar to that made on price expectations, has to be imposed to get stability. Again, that assumption is wide enough to accommodate perfect foresight and also naïve expectations.

A word of caution must be given, however. While it is obviously interesting to depart radically from the assumption that agents always

believe that they are in equilibrium and instead understand what is happening, there is a sense in which too radical a departure is self-defeating. Thus, if agents have perfect foresight about everything, and, in particular, if they have perfect foresight about transaction constraints, then they will never attempt to violate those constraints and (since they know them correctly), nothing will ever move. Every point in such a case will be an equilibrium of the dynamic process, although it will not generally be a Walrasian competitive equilibrium. The essence of disequilibrium is that somebody attempts to take an action which cannot be fulfilled.

Even if it makes no sense to allow agents perfect foresight as to transaction constraints, however, it is obviously desirable to allow them to understand that there are transaction difficulties as well as to believe that prices will change. In that sense, this paper may represent an advance. It is important to recognize, nevertheless, that permitting such awareness in the way done here by no means solves the problem of allowing consciousness of disequilibrium in a fully satisfactory way. Even allowing agents to alter their expectations in sensible ways does not permit them to take into account the fact that their expectations may be wrong. Simply put, agents in the present model always behave as if they lived in a world of certainty. While we can now allow them to open their eyes to the fact that prices do change and that completion of transactions may be difficult, we do not allow them any awareness of the equally persistent fact that (except in the case of perfect foresight) they are often wrong about prices and about transaction completion.

I do not see the way towards a satisfactory solution here. Micro-economic theory is primarily an equilibrium subject. We know very little about the theory of individual behavior in disequilibrium. Further, a

full-dress treatment of behavior under uncertainty in a disequilibrium situation strikes me as too complex for incorporation into this sort of model at least in the present state of our knowledge. Lacking one, we are forced to consider either allowing agents to be conscious of disequilibrium without uncertainty, as in the present paper, or restricting their behavior to certain rules of thumb (which is one way of looking at Fisher [17]).

One such rule of thumb which can be thought of as incorporated in the present paper is that large actions ought not to be undertaken in pursuit of very small gains. Hence one way of regarding the role of transaction costs in the present analysis is as an inadequate substitute for a full-dress incorporation of uncertainty.

The aspect of disequilibrium behavior other than uncertainty which is still largely ignored in the present paper is that of how prices change. Although it is possible (and indeed desirable) to think of price setting as done by individual agents, with commodities distinguished by the name of the agent who sells them, as in Fisher [11], it remains true that the behavior of such price-setting agents (left implicit in the present paper) is not generally consistent with optimal price-setting behavior in a disequilibrium context when a certain amount of monopoly power may be inevitably present. Such issues remain very difficult.⁴

4. See Rothschild [29] and Fisher [10, 12].

The fact that allowing for disequilibrium consciousness as is done here does not solve everything in such models does not mean that it does not solve anything, however. As it turns out, with such disequilibrium

consciousness it is possible to get round some other problems that have vexed earlier work.⁵

5. There are also some technical improvements in the present paper (convergence in norm rather than weak* convergence is proved for prices; some earlier errors are corrected). These are not especially related to disequilibrium consciousness, however.

The first of these is as follows. In previous models along these lines, equilibrium has meant an exhaustion of trading opportunities. In other words, when equilibrium is reached, all trades that can be made at the equilibrium prices have been made. Agents then go and carry out consumption and production plans, delivering on promises paid for before equilibrium. Trading itself stops, however. While the fact that equilibrium is only approached asymptotically may make this a bit less awkward than it sounds, it would obviously be preferable to include the possibility that, after equilibrium is reached, trading continues to take place at previously foreseen prices, so that the reaching of equilibrium need not render markets and prices redundant by collapsing the economy into a state of autarchy. The dropping of the naïve price expectations assumption allows us to do this.

More important than this problem (and, indeed, largely producing it) is what I have elsewhere called the "Present Action Postulate."⁶

6. [14, 15, 16].

In previous studies it has been necessary to assume that unfulfilled

plans result in immediate actions. A consumer with a positive excess demand for toothpaste must actually attempt to buy toothpaste (assuming he has money with which to do so). It is not hard to see that some such assumption is generally required in all of the traditional stability literature. If prices are to be driven by excess demands, then excess demands which remain gleams in the eye of the demander cannot affect prices and the system will bog down. Nevertheless, the requirement that present action be taken can be very awkward. If there are no futures markets, for example, the fact that I expect to require toothpaste ten years hence is made to propel me into the spot market for toothpaste even if I am having some liquidity problems and am trying to buy commodities needed sooner. Even with a complete set of futures markets the problem is only partially alleviated.⁷ Here, my need for toothpaste ten years hence

7. The case without futures markets is treated in Fisher [14]; the case with complete futures markets in Fisher [16].

sends me immediately into the futures market for toothpaste. Since I must pay for futures purchases today, this is only somewhat more plausible.

It turns out that the objectionable nature of this sort of assumption disappears when price expectations involve expectations of change. The reason for this is as follows. When current prices are expected to persist forever, there is no incentive to go out and buy toothpaste before one needs it to brush ones teeth. If that date is ten years hence, then anytime in the next ten years will do, since one expects the toothpaste always to be on sale and available at the same (current) price. It thus seems objectionable to force agents to act immediately (particularly if

liquidity considerations are only introduced as an afterthought). When prices are expected to change, however, agents cease to be indifferent to the date at which a certain transaction is consummated. Demands must then be indexed not merely by the date of the futures market involved but also by the date at which the purchase attempt is to be made. Thus, for example, the demand for 1989 toothpaste to be purchased in 1985 has to be treated separately from the demand for 1989 toothpaste to be purchased in 1986. Once one has done this, however, the objectionability of (or, if you like, the need for) the Present Action Postulate disappears. One can indeed assume that all demands are acted on presently, but this is vacuous since acting on a demand for 1989 toothpaste to be purchased in 1985 requires doing nothing at all in 1979.

There are other, more minor gains from allowing disequilibrium consciousness. If commodities are distinguished by date, as is natural to do, then production and consumption can only take place out of equilibrium if we suppose that some commodity dates are passed during the adjustment process. Since there are no "pasts" markets, this means that trading ceases in commodities whose dates are passed. If agents always believe that they can consummate their transactions, such cessation of trading can produce a rude awakening and awkward discontinuities in excess demand functions, since the same commodity with different dates can trade at different prices. Out of equilibrium it is not guaranteed that arbitrage will successfully bring the prices for the same commodity in two time periods together as the time periods join up.

One way to handle this is to have an individualistic model of price adjustment in the background. Essentially, this amounts to having each commodity have its prices at all dates in charge of a specialist in whose

interest it is to preserve continuity. This is the approach taken in Fisher [16] and will be preserved here.⁸ The difficulties involved,

8. See also [11]. For lengthier discussion of the dated commodities problem, see [14] and [15]. Transaction costs may prevent any offers being made for certain dates. We assume that prices continue to be quoted, however.

however, largely disappear if we allow disequilibrium consciousness and drop the silly assumption that agents always expect to complete their transactions right up to the closing bell. If agents are aware of disequilibrium, then we can assume that they expect it to get harder and harder to complete transactions in a particular dated commodity as its date approaches. (This can be expressed either in terms of transaction costs or in terms of transaction constraints.) This will prevent the close of trading from coming as a surprise and causing discontinuities. While this is not strictly needed if prices are assumed to be continuous in commodity dates, it seems a natural thing to do.

Next, the fact that there can be different expectations about prices means that there will be different opinions as to the future profits of firms. This makes it natural to introduce a market for shares, a feature missing from past versions. (It also leads to some interesting problems.) Interestingly enough, it turns out that the crucial thing which makes a market for shares a useful part of the model is not so much the possibility of differing profit expectations but the fact that agents expect prices to change. Even if all such expectations are the same and, indeed, even if they are all correct, the changing nature of relative prices

creates a demand for transactions in shares essentially because the marginal rate of substitution between money and other goods is different at different times.

As this suggests, money plays an important role in this model (although with uncertainty left out the monetary theory involved is still rudimentary). As in earlier papers, following Arrow and Hahn [1], it is assumed that transactions take place only for money. Money is an undated commodity so far as notation is concerned, but, as we shall see, not in actuality.⁹ The price of money is unity. Unlike earlier papers,

9. This is not to say that agents do not care when they have money.

however, it is not necessary to allow agents only as an afterthought to take account of the fact that demands must be backed up with money. Rather they can now be allowed to do this as part of their optimization processes, requiring them never to plan to make purchases while short of money.¹⁰

10. Since money is a stock and purchases are flows, this makes it unnecessary to worry about rationing of money over demands.

It is still necessary, however, to assume that agents never in fact find themselves attempting to make such purchases (which is a little more palatable if they are allowed to plan about this in advance). In the present model, such a No Bankruptcy assumption becomes rather similar to the general (and strong!) assumption which so far appears crucial in models with production and consumption, namely, that agents always have enough goods on hand or delivered to them to enable them to carry

out current consumption and production plans. In some cases, this means assuming that outputs whose delivery has been contracted for in fact get delivered; in the case of money it amounts to assuming that some current positive purchase plan can be carried out (provided a seller can be found). In both cases it means assuming that plans are not so wildly wrong that activities must abruptly cease altogether for want of an essential input.

Note also that, as with the No Bankruptcy assumption on money, the assumption that current consumption and production plans can be carried out is somewhat more attractive when agents are allowed disequilibrium consciousness and are aware that closing dates for delivery do come. We shall insist that agents not make unbounded delivery commitments and, indeed, shall require that the extent to which they plan to be short in any commodity go to zero as the date on that commodity approaches. This will reduce, but not prevent the possibility that they actually will be caught short. How to handle that possibility is a problem for later work.

The plan of the paper is as follows. After a discussion of discounting and of the choice of space in which to work, I examine the disequilibrium behavior of the individual firm and the individual household. Since this turns out already to be fairly complicated, the treatment of transaction costs and transaction constraints is postponed to a later section of its own. I then go on to discuss the interconnections between firms and households, the market for shares having already been introduced in the discussion of households, and consider closure equations and Walras' Law. With the description of how the economy looks at any one point in time completed, it is then possible to go on to a considera-

tion of the adjustment process and, in particular, to a discussion of the way in which expectations get revised. After that is done the stability results are obtained.

Just what do those stability results turn out to be? Before previewing them it may be wise to step back for a moment and consider what one can expect. Clearly, in a model as rich as this one in allowing disequilibrium awareness, one can no longer expect to pursue the traditional aim of proving stability no matter what. There is no way in which the model can be stable if agents can always wake up in the middle of the night with a new idea on which to act. It makes no difference whether such ideas are realistic or mistaken, the perception and pursuit of a new opportunity, however mistaken, will keep things moving.

The principal stability result of the present paper is that if such new perceptions are ruled out, then the economy converges to equilibrium. Somewhat more precisely, if, after some finite time, expectation revision is always such as to leave agents no better off in terms of their anticipated profits or utilities, then the adjustment process is globally stable. I shall refer to such a circumstance as one of "No Optimistic Surprise." Note that it includes the case of perfect foresight.¹¹

11. It also generalizes the Hahn Process assumption for naïve expectations. See Hahn and Negishi [20], Arrow and Hahn [1], and Fisher [11, 13, 14, 15, 16, and 18].

I think the proper way to look at this result is not as stating that No Optimistic Surprise provides a sufficient condition for stability. Rather it is as stating that Optimistic Surprise is necessary for instability.

In effect, what is being said is that if all that is happening is that old opportunities are disappearing through arbitrage, then the system will be stable. It is the perception of new opportunities that keeps the system moving.

Such a result may be new in formal stability literature, but it is not new in economics. Aside from the rational expectations flavor which stability under No Optimistic Surprise has as old opportunities are arbitrated away, the result that new opportunities are what keeps the system from approaching equilibrium must bring to mind the name of Schumpeter.¹²

12. See Schumpeter [30, 31].

I now proceed to the formal model.

2. Prices, Bonds, Discounting, and Choice of Space

There are n ordinary commodities in addition to money. Each of the ordinary commodities is (in principle) distinguished by date. As of time, t , the vector of prices of θ -dated commodities is given by $\hat{P}(\theta, t)$. Here (and, generally later), θ is in $[\underline{0}, \infty)$ with $\hat{P}(\theta, t)$ for $\theta \leq t$ being fixed at the closing prices which obtained at θ ; in other words, $\hat{P}(\theta, t) = \hat{P}(\theta, \theta)$ for all $t \geq \theta$. $\underline{0}$ is an arbitrary initial date. $\hat{P}(\theta, t)$ is assumed continuous in θ and differentiable in t . Derivatives of any function with respect to t will be denoted by dots.

Agents have expectations about prices. We let $\hat{p}(\theta, v, t)$ (agent subscript omitted) denote the vector of prices of θ -dated commodities which, at time t , the agent expects to encounter at time v . $\hat{p}(\theta, v, t)$ is assumed differentiable in t and, for given t , has all the properties of $\hat{P}(\theta, t)$. In other words, $\hat{p}(\theta, v, t)$ is continuous in θ and differentiable in v . Furthermore, $\hat{p}(\theta, v, t) = \hat{p}(\theta, \theta, t)$ for all $v \geq \theta$. We further assume that the agent knows correctly all present and past prices so that $\hat{p}(\theta, v, t) = \hat{P}(\theta, v)$ for all $t \geq v$.¹

1. This is weak consistency in the sense of Turnovsky and Burmeister. See [32], for example. I am indebted to J. Cuddington for this reference.

Now, it will be an essential feature of the model that agents plan as to the timing of their transactions. This means that we must consider the way in which they discount the future. In essence, this means considering the way in which future money is discounted, since such matters as time preference for households and capital productivity for firms will automatically be taken care of in the specification of utility and production functionals below.

I shall proceed as follows.² Among the n ordinary commodities there

2. I am indebted to Kenneth Arrow for calling my attention to this area and to Eric Maskin for helpful discussions of the issues involved.

is one called "bonds". The sale of a θ -dated bond is a legally binding promise to pay one dollar at date θ . It is a way of transferring money from the future. Since money (which is otherwise treated as undated) can be costlessly stored, bonds will ordinarily sell at a discount reflecting a forward interest rate.

Given the existence of bonds, agents must discount future receipts, the promise of the receipt of a dollar at time θ being the equivalent of possession a θ -dated bond. Such discounting appears both in the "money constraints" below in which agents plan always to have non-negative money balances and in the profit calculations of firms who measure earnings in terms of their present money equivalent.

Now, I have said that a dollar of θ -dated receipts must be treated by the agent as the equivalent of holding a θ -dated bond. Does that mean that the agent discounts such receipts back to t by using the interest rate implicit in the θ -bond price holding at t ? Out of equilibrium, it is easy to see that this need not be the case. If it were, then agents would never find any speculative gains in buying or selling bonds even out of equilibrium. Further, it is not hard to show that in such a case, an agent with perfect foresight, able to fulfill his optimal plan would change that plan merely because bond prices changed in a way in which he had fully taken account of in the first place. This would occur because his relative discounting of future periods would change. Obviously, this makes no sense.

Fortunately, one does not need to look very far to discover what the proper discount rate is. Out of equilibrium, agents have a choice of ways in which to transfer money between time periods. Thus, to transfer money from t to some $v > t$, an agent can buy a v -dated bond at t or he can buy a v' -dated bond at t and a v -dated bond at v' , for some v' between t and v . Obviously, there is no need for him to deal in only two short-term bonds instead of one long-term bond; indeed, since future commodities, including bonds, are continuously dated, he can deal continuously. Any such method implies a way of discounting the future.

Which such implied discount rate is correct? Any of them but the limiting continuous compounding of short rates leads to the anomalous behavior discussed above. Only the continuous compounding of short rates avoids having optimal programs changed when they remain feasible and all expectations are correct. Moreover, only that version makes it potentially profitable to deal on all bond markets of non-zero length if such markets provide opportunities for arbitrage.

It is important to emphasize, however, that this choice of discount rate does not mean (as it would in equilibrium) that agents necessarily choose between present and future consumption or production applying that rate. If there are profitable speculative opportunities (including the opportunity for arbitrage in later-dated bonds), then agents will find that the shadow price ratio at which present and future actions are traded includes the opportunity costs of foregoing such speculations. This will appear in the first-order conditions in later sections.

We can now formally derive the relevant discount rate from the bond prices. Let $\hat{o}(\theta, v, t)$ (included in $\hat{p}(\theta, v, t)$) be the price which, at t , the agent expects to encounter at v for θ -dated bonds. Divide the

interval from t to θ into k equal parts of length Δv , so that $\theta = t + k\Delta v$.

If money is transferred from θ to t by a series of k purchases of bonds of length Δv , then the implied discount factor, $\hat{\rho}(\theta, t; \Delta v)$ is given by:

$$(2.1) \quad \log \hat{\rho}(\theta, t; \Delta v) = \sum_{j=1}^k \log \hat{\sigma}(t + j\Delta v, t + (j-1)\Delta v, t) =$$

$$\sum_{j=1}^k \frac{\log \hat{\sigma}(t + j\Delta v, t + (j-1)\Delta v, t) - \log \hat{\sigma}(t + (j-1)\Delta v, t + (j-1)\Delta v, t)}{\Delta v}$$

using the fact that $\hat{\sigma}(v, v, t) = 1$ for all v . Passing to the limit, and denoting the limiting discount factor by $\hat{\rho}(\theta, t)$,

$$(2.2) \quad \log \hat{\rho}(\theta, t) \equiv \lim_{\Delta v \rightarrow 0} \log \hat{\rho}(\theta, t; \Delta v) =$$

$$\int_t^\theta \frac{\partial \log \hat{\sigma}(a, v, t)}{\partial a} \bigg|_{a=v} dv = \int_t^\theta \frac{\hat{\sigma}_1(v, v, t)}{\hat{\sigma}(v, v, t)} dv =$$

$$\int_t^\theta \hat{\sigma}_1(v, v, t) dv$$

where the subscript denotes differentiation in the obvious way, and we have used again the fact that $\hat{\sigma}(v, v, t) = 1$.

Now, in this definition, we have used the instantaneous discount rates which the agent expects to encounter after t ; it turns out generally to be convenient to discount back to a fixed date, 0. In so doing, we merely use the actual bond prices prevailing at each date before t . Thus, letting $\hat{\sigma}(\theta, v)$ denote the actual price of θ -dated bonds which prevailed at v , the discount rate from t back to 0 for our purposes will be given by

$$(2.3) \quad \log \bar{\rho}(t, 0) = \int_0^t \sigma_1(v, v) dv$$

and the discount rate from any later date, θ back to 0 by

$$(2.4) \quad \rho(\theta, 0) \equiv \bar{\rho}(t, 0) \hat{p}(\theta, t).$$

Notice that $\rho(\theta, 0)$ is a combination of actual and expected rates. We shall take $\rho(\theta, 0) = \bar{\rho}(\theta, 0)$ for $\theta \leq t$.

It turns out to be notationally convenient for most of the paper to work with all prices (and later all revenue streams) discounted back to 0. Accordingly, define:

$$(2.5) \quad P(\theta, t) \equiv \hat{P}(\theta, t) \rho(t, 0); p(\theta, v, t) \equiv \hat{p}(\theta, v, t) \rho(v, 0).$$

Note that $p(\theta, v, t)$ includes expectations as to discount rates.³ I

3. I hope it is not misleading that the prices without the hats are the discounted ones; to reverse it would mean carrying the hat notation everywhere.

shall keep on calling the discounted prices "prices" from time to time except where it is important to distinguish them from undiscounted ones. (Of course one can always read as though there were no discounting.)

Now, so far nothing has been said about the space in which all this is supposed to be happening. I have deliberately postponed that discussion until now because it turns out to involve the discount rate in an essential way. It is, of course, natural to take consumption, production and other such profiles as lying in a normed linear space and to take prices as lying in the normed dual of that space. But what space? There are some issues here which need discussion.

In the first place, although such profiles (which we shall denote generically by $z(., t)$ in this section)⁴ are conveniently taken for most

4. Some of them will later involve a notation such as $z(., v, t)$, but that is irrelevant for purposes of this section, and we shall ignore it.

purposes as defined over $[\underline{0}, \infty)$, with 0 an arbitrary, but fixed starting date, this is not a convenience when considering convergence. This is so because the mere passage of time will fix more and more of the early part of such profiles with only the part from t on remaining free. It would obviously be silly to adopt a norm in which that fact alone implied convergence. This consideration will be very restrictive, however, if we insist on counting the fixed part of such profiles, the part in $[\underline{0}, \underline{t}]$.

Clearly, the natural thing to do is to count only the free tail, the part in $[\underline{t}, \infty)$.⁵ We can readily do this by a change of variables

5. This was not done in Fisher [16]. The discussion on this point there is misleading and irrelevant; it ought properly to be replaced with the equivalent of the present one. In the context of that paper, the present discussion would imply that all commodity profiles are in L_1 , but in fact there is no reason not to interpret prices in that paper as discounted prices and apply the present discussion directly. The theorems of that paper remain true with such an interpretation, so the matter is not of great importance, although it is as well to get it right. The essential difference between that paper and the present one is neither the choice of space nor even the presence or absence of discounting (which could be incorporated in the previous paper by appropriate interpretation) but rather the consciousness or lack of consciousness of disequilibrium.

(used only in the present section) by measuring commodity dates, θ , from t rather than from 0. In this notation, all commodity profiles from t onwards lie in $[\bar{0}, \infty)$. I shall denote such profiles by $\tilde{z}(\cdot)$, suppressing the t argument. Similarly, I shall denote the correspondingly redated tail of the price profile,⁶ discounted back to time, t , rather than to

6. There are actually many such profiles corresponding to the differing expectations of different agents; I ignore this in the present discussion.

time 0, by $\tilde{p}(\cdot)$.

Now, much of the theory below will only make sense if the value of commodity profiles discounted back to t is finite. Since we do not wish this necessarily to imply that t -discounted prices approach zero, we must require that the integral of any commodity profile, $\tilde{z}(\cdot)$, itself discounted back to its beginning (t) be bounded.

All of this suggests that the commodity profiles be assumed to lie in a space in which such integrals are bounded. It is mildly inconvenient to do this directly, however, because the discount rates involved vary from agent to agent and from time period to time period (although we shall later show their convergence to a common profile for all agents).⁷

7. But not for all time periods. There is no reason why equilibrium cannot occur with different (but correctly foreseen) instantaneous interest rates for different moments in time, depending on the (possibly changing but correctly foreseen) time preferences and technological opportunities of agents.

Accordingly, I shall adopt a somewhat more restrictive version than is strictly necessary.

Thus, using the notation introduced earlier in this section, and remembering the redating of θ for present purposes, let $-\rho$ be the maximum over all agents of $\text{Sup } (\hat{\sigma}_1(v, v, t))$. Then $e^{-\rho\theta} \geq \rho(\theta, t)$ for all agents and all t and θ (See (2.2).), so that all agents always discount the future at a rate at least as great as the constant rate ρ . Thus convergence of the improper integral $\int_0^\infty |\tilde{z}(\theta)| e^{-\rho\theta} d\theta$ for some profile $\tilde{z}(\cdot)$ implies convergence of the improper integral $\int_0^\infty |\tilde{z}(\theta)| \hat{\rho}(\theta, t) d\theta$ for that same $\tilde{z}(\cdot)$ and for all discount factor profiles used by agents. Accordingly, the commodity profiles, $\tilde{z}(\cdot)$, will be taken to lie in the space of functions defined on $[\underline{0}, \infty)$ with norm $\int_0^\infty |\tilde{z}(\theta)| e^{-\rho\theta} d\theta$. That space will be denoted by Z .

One more remark before proceeding. It is not very interesting if Z turns out to be L_1 (and also it is technically inconvenient) so that I shall assume $\rho > 0$. This amounts to assuming that expected bond prices are always such as to bound instantaneous discount rates away from zero.

Now, denote the normed dual of Z by Z^* . The t -discounted price profiles, $\tilde{p}(\cdot)$, will be taken to lie in Z^* . This means that the norm of any such t -discounted price profile will be:

$$(2.6) \quad \text{Norm } (\tilde{p}(\cdot)) = \sup_{\substack{\tilde{z}(\cdot) \in Z \\ \tilde{z}(\cdot) \neq 0}} \frac{\left| \int_0^\infty \tilde{p}(\theta) \tilde{z}(\theta) d\theta \right|}{\int_0^\infty |\tilde{z}(\theta)| e^{-\rho\theta} d\theta} = \sup |\tilde{p}(\theta)| e^{\rho\theta}.$$

The last step in (2.6) can perhaps be seen most conveniently by observing

that, since $(\tilde{z}(\theta)e^{-\rho\theta})$ forms a profile which lies in L_1 , and all such profiles can be formed in this way, $(\tilde{p}(\theta)e^{\rho\theta})$ must form a profile in L_∞ .⁸

8. We shall take the prices continuous in θ so that there is no need to distinguish between the supremum and the essential supremum. See Luenberger [26, p. 32].

The meaning of this result is that the appropriate norm for t-discounted prices is the supremum of the undiscounted prices which would correspond to them at the constant interest rate ρ . This is not quite the same as the supremum of the actual expected undiscounted prices (the difference occurring because of the inconvenience avoided by choosing a norm for Z which is independent of time period and agent). However, it is plain that, because of the definition of ρ , discounted prices in Z^* will be bounded if the supremum of the actual expected undiscounted prices is bounded for every agent.

We can now return to discounting prices back to time 0.

3. Behavior of Firms

Firms will later be subscripted f , but I shall omit the subscript wherever possible. Further, I shall use a parallel notation for firms and households to avoid an even greater proliferation of notation than occurs below.

For convenience, firms are owned only by households, not by other firms. They only raise new capital by acting on the bond market; new equity capital is not treated. Capital is implicitly raised from shareholders by retaining earnings, however.

The firm has a program of planned (or, in the case of dates already passed, actual) sales and purchases of ordinary commodities. It is convenient to measure sales negatively and purchases positively. We denote by $r(\theta, v, t)$ the n -vector of purchases of θ -dated commodities which, after trade at t , the firm plans to make at v . Obviously it makes sense to define $r(\theta, v, t) = 0$ for $v > \theta$. Actual purchases made at v will be denoted by $\bar{r}(\theta, v)$ with a similar convention for $v > \theta$.

We can now write the discounted profits expected by the firm, $\pi(t)$, as:

$$(3.1) \quad \pi(t) = - \int_0^\infty \int_t^\infty p(\theta, v, t) r(\theta, v, t) dv d\theta + \bar{\pi}(t)$$

where the product in the integrand is to be taken as an inner-product (and similarly throughout) and $\bar{\pi}(t)$ denotes profits already achieved at time t . Thus:

$$(3.2) \quad \bar{\pi}(t) = - \int_0^\infty \int_0^t P(\theta, v) \bar{r}(\theta, v) dv d\theta + \bar{\pi}(0).$$

Now what are the constraints on the firm (apart from transaction constraints which we consider in a separate section)? We let $y(v, t)$ denote the vector of ordinary commodities which, at t , the firm plans to use as inputs in its production process.¹ Note that there is no

1. It is convenient, although not essential to assume that money plays no direct role in the production process either as input or as output.

issue as to the dates on such inputs; only v -dated commodities can be used as inputs at v . For $v \leq t$, $y(v, t)$ represents actual inputs made at v .

As a result of its input activities, the firm at t expects to have at each time, θ , a flow of outputs of θ -dated goods.² That flow is given

2. Natural endowments are readily handled within the same notation. So are perishable inputs.

by a production functional, $\phi(y(., t), \theta, t)$, which is assumed continuously Fréchet differentiable. A few more words about this are in order.

In the first place, outputs at θ only depend on inputs up to θ . Moreover, it is sensible to suppose that the outputs of each commodity depend only on the inputs devoted to the production of that commodity rather than on total inputs as the notation would indicate. Indeed this is so; however, there is no point in burdening the notation with it. Instead, the argument, t , in part indicates that the way in which the inputs have been and are planned to be used depends on the history of

actions taken up to t and the expectations of the firm at t . The sub-optimization decisions of the firm are kept implicit.

This is a different point from the erroneous one that there is something wrong because the same inputs keep on producing outputs regardless of what is taken out of the process in earlier periods. Only θ -dated goods can be outputs at time θ . As we shall formally state in a moment, θ -dated goods which are not sold but remain on hand at θ cannot remain θ -dated goods. This being so, we may as well think of them as automatically becoming inputs into the production process, even if that process becomes in part the trivial one of storing commodities so that their dates change. Hence later outputs do depend on earlier output decisions. They in effect depend on how much of earlier outputs remain as inputs. Perhaps the easiest way to think about this is to take $\phi(y(\cdot, t), \theta, t)$ as denoting the availability of outputs of θ -dated goods at θ .

While the production process can only produce θ -dated goods at θ , the firm can acquire such goods at other times through purchase. Thus, let $x(\theta, v, t)$ be the stock of θ -dated goods which the firm, at t , expects to have acquired through purchase and have on hand at $v \leq \theta \leq t$. For $v \leq t$, the actual stock on hand will be denoted by $\bar{x}(\theta, v)$. We shall think of trade as occurring instantaneously or outside of time, so that $x(\theta, v, t)$ and $\bar{x}(\theta, v)$, like $r(\theta, v, t)$ are to be evaluated after trade at v . Thus:

$$(3.3) \quad x(\theta, v, t) \equiv \bar{x}(\theta, t) + \int_t^v r(\theta, a, t) da \quad v \geq t$$

and

$$(3.4) \quad \bar{x}(\theta, t) \equiv \bar{x}(\theta, 0) + \int_0^t \bar{r}(\theta, a) da.$$

Now, as stated, θ -dated commodities on hand after trade at θ become inputs. Formally:

$$(3.5) \quad y(\theta, t) = \phi(y(\cdot, t), \theta, t) + x(\theta, \theta, t).$$

We have now described the technological constraints on the firm. If these were the only constraints, however, certain problems would arise. In particular, for certain profiles of expected prices, the firm would want to sell infinite amounts of future commodities short expecting to cover when prices change. This would not only directly cause technical problems but would also lead to situations where promised goods could not be delivered. Aside from the difficulties which that would cause for the defaulting seller, it could also mean that input and production plans cannot be carried out not because commodities have not been bought in time but because the purchases turn out suddenly to be phantom ones. It is hard in any case to assure that this does not happen when transactions may not be completed, and we shall have to assume that it does not. There is no need, however, to suppose that firms (or households) get themselves (and others) into such a fix by deliberately going infinitely short. Hence we shall impose a bound on short sales. That bound may be thought of either as a legal requirement or as a self-imposed limitation which the firm obeys because of the uncertainty not explicitly modelled. In the latter context it makes most sense where the firm realizes it may not be able to complete its transactions and clear its position by purchasing a great deal just before delivery. I return to transaction constraints below.

It would be possible to make the bound in question an over-all bound on the value of short sales, but I have chosen to do it by placing

specific limits on the short sales of each commodity and assuming that this suffices to bound the discounted value of all short sales.³

3. It would be possible to make the bounds a function of overall debt or of other short sales but there seems little point in such complexities.

Note, in this connection, that the sale of a θ -dated bond is treated as a sale of a θ -dated commodity so that there are limits on the ability of the firm to borrow. Since bonds do not enter the production process, long or short positions in θ -dated bonds must be cleared by trading, including the repayment of such bonds at θ by repurchasing them at a price equal to unity. Bond redemption is thus implicitly enforced in the short constraints.⁴

4. Since purchases are flows this may imply an infinite repayment rate at θ . I shall not trouble about this since transaction costs, later imposed, can make it optimal to repay over an interval rather than at a point.

It is obviously sensible to require that the bounds on short sales of θ -dated commodities approach zero as θ comes nearer. Accordingly, I assume that there exists an n -component vector, $\varepsilon(\theta - v) \geq 0$,⁵ continuous

5. This only matters for $\theta \geq v$. I use the usual convention for vector inequalities, i.e., $x \geq y$ means $x_i \geq y_i$, and $x \geq y$ means $x \geq y$ but $x \neq y$.

in $(\theta - v)$ and with $\varepsilon(0) = 0$, such that:

$$(3.6) \quad x(\theta, v, t) + \phi(y(\cdot, t), \theta, t) + \varepsilon(\theta - v) \geq 0 \quad \text{for all } \theta \geq v \geq t.$$

Note that the firm is entitled to count at each v its anticipated output of θ -dated goods evaluating its position as though its purchases of such goods had to stop.⁶

6. Note also that the case in which short sales are simply not permitted is included.

Now, just as the firm is not allowed to plan to be too short in ordinary commodities, so we assume that it does not plan to be short in money. This is because transactions can only be carried out for money.

As of time t , the firm has announced dividends. We denote by $\bar{G}(t)$ the discounted value of total dividend payments already paid by the firm up to and including t . It is natural to assume:

$$(3.7) \quad \bar{G}(t) \leq \bar{\pi}(t),$$

and, as we shall see, in general the strict inequality may hold out of equilibrium (and the equality only asymptotically in equilibrium) since the firm has to retain earnings for working capital. I shall have more to say about dividends below.

The constraint that the firm must plan to have a non-negative money stock can now be written as:

$$(3.8) \quad \bar{\pi}(t) - \bar{G}(t) - \int_0^\infty \int_t^v p(\theta, a, t) r(\theta, a, t) da d\theta \geq 0 \quad \text{for all } v \geq t.$$

Note that this makes firms take into account the fact that purchases must be made with money.

It may seem a little odd to state (3.8) in terms of discounted rather than undiscounted prices, so that it, in effect, states that the present value of all commitments planned through date v shall be non-negative. After all, the firm will have to pay for its purchases when they occur in what will then be the undiscounted prices. This is misleading. It is true that the firm will have to pay in undiscounted prices, but it will be paying in what is now future money. It is only notationally that money is undated. Plainly, in any sensible program, current money which is in excess will be moved forward at least at the instantaneous discount rate. Hence the optimal program which satisfies (3.8) will have a pattern of bond sales and purchases which match the availability of money to the times at which it is needed. To put it another way, the very reason for discounting discussed in the previous section stems from the fact that all programs will move money forward as described. It would be a mistake to ignore that fact here.⁷

7. Since the optimal program satisfying (3.8) will also satisfy a similar constraint in current prices it might still be thought that such a constraint is the appropriate one. This is a somewhat subtle issue because it bears mainly on the question of when the appropriate constraint is and is not binding. Aside from the reasoning given in the text as to the point of discounting in the first place, one can observe that (3.8) clearly gives the right results as to speculative behavior derived below; a similar constraint in current prices does not.

For mnemonic convenience, I shall refer to (3.8) as the "money constraint" and to the combination of (3.3) and (3.6) as the "short constraints".

Finally, it is convenient to require explicitly that planned inputs, $y(\theta, t)$, be non-negative. This is implicit in (3.5) and (3.6), but the latter constraints are best thought of as assuring the non-negativity of the stock of goods available for inputs rather than of inputs themselves.⁸

8. Without an explicit non-negativity constraint on inputs one has to be careful as to whether certain integrals of Lagrange multipliers are automatically zero when upper and lower limits of integration coincide. (Of course the Lagrange multipliers in these various constraints are related.)

Another way of putting it is to assume free disposal and replace (3.5) with an inequality stating that inputs cannot exceed available stocks. Then one needs another inequality preventing inputs from being negative. One can manage without this but it is awkward to do so.

The firm maximizes discounted profits⁹ (3.1) subject to (3.3), (3.5),

9. As P.A. Diamond has pointed out to me, the assumption that firms maximize present discounted value is not automatic if the firm and its stockholders have different expectations. In the present model, there is a securities market and, I assume, a variety of potential stockholders. Firms maximize present discounted value according to the expectations of their managers, and stockholders who disagree can sell off their holdings in accordance with the optimal programs for households developed in the next section.

(3.6), (3.8), and the input-non-negativity constraint. Substituting (3.3) into (3.5) and (3.6), the appropriate Lagrangian is given by:

$$\begin{aligned}
 (3.9) \quad L(t) = & \bar{\pi}(t) - \int_0^\infty \int_t^\infty p(\theta, v, t) r(\theta, v, t) dv d\theta \\
 & - \int_t^\infty \lambda(\theta, t) \{y(\theta, t) - \phi(y(\cdot, t), \theta, t) - \bar{x}(\theta, t) - \int_t^\theta r(\theta, v, t) dv\} d\theta \\
 & + \int_t^\infty \int_t^\theta \mu(\theta, v, t) \{\bar{x}(\theta, t) + \int_t^v r(\theta, a, t) da + \phi(y(\cdot, t), \theta, t) + \varepsilon(\theta - v)\} dv d\theta \\
 & + \int_t^\infty \mu_0(v, t) \{\bar{\pi}(t) - \bar{G}(t) - \int_0^\infty \int_t^v p(\theta, a, t) r(\theta, a, t) da d\theta\} dv \\
 & + \int_t^\infty \gamma(\theta, t) y(\theta, t) d\theta.
 \end{aligned}$$

Here, $\lambda(\cdot, t)$, $\mu(\cdot, \cdot, t)$, $\gamma(\cdot, t)$ and $\mu_0(\cdot, t)$ are Lagrange multiplier functions, the first three being vectors.

The first-order conditions for a maximum are:¹⁰

10. It is obvious that the solution to the maximum problem as so far presented will generally be a "bang-bang" one and need not be continuous in the prices or even unique in all respects. This will be true of households as well. I shall get rid of these problems below by introducing transaction costs and shall ignore them for the present.

$$(3.10) \quad \lambda(\theta, t) + \int_v^\theta \mu(\theta, a, t) da = p(\theta, v, t) \left\{ 1 + \int_v^\infty \mu_0(a, t) da \right\}$$

$$\theta \geq v \geq t$$

and

(3.11)

$$\lambda(v, t) - \gamma(v, t) = \int_v^\infty \left\{ \lambda(\theta, t) + \int_v^\theta \mu(\theta, a, t) da \right\} \phi_{y(v, t)}(y(\cdot, t), \theta, t) d\theta$$

$$v \geq t$$

where the subscript denotes the value of the Fréchet derivative at v .¹¹

11. It is best to be precise about the notational convention used.

The Fréchet derivative of $\phi(y(\cdot, t), \theta, t)$ would be denoted $\phi_y(y(\cdot, t), \theta, t)$.

It is an entire function defined over v . It is convenient to have a notation for the value of that function at a specific v , this is given by

$\phi_{y(v, t)}(y(\cdot, t), \theta, t)$. This notation which will be abbreviated $\phi_{y(v, t)}$

wherever possible is mnemonic because it reminds us of the fact that its value is the limit of the derivative with respect to a constant added to

$y(., t)$ from $v - \varepsilon$ to $v + \varepsilon$ as ε goes to zero. A similar convention is used for utility and storage functionals below.

Since certain of the integrals in (2.10) and (2.11) will occur frequently in the following discussion, it will be convenient to adopt a notation for them. Accordingly, we define:

$$(3.12) \quad I(\theta, v, t) \equiv \int_v^\theta \mu(\theta, a, t) da \quad v \leq \theta$$

and

$$(3.13) \quad J(v, t) \equiv \int_v^\infty \mu_0(a, t) da.$$

The separate components of $I(\theta, v, t)$ will be denoted by subscripts.

We now proceed to a discussion of the implications of these conditions.

Theorem 3.1:

$$(3.14) \quad \int_v^\infty p(\theta, v, t) \phi_{y(v, t)} d\theta \leq p(v, v, t)$$

with equality holding for those commodities which the firm actually plans to use as inputs at v .

Proof: Substituting from (3.10) into (3.11) and also evaluating (3.10) at $\theta = v$ yields:

$$(3.15) \quad p(v, v, t) \{1 + J(v, t)\} - \gamma(v, t) = \{1 + J(v, t)\} \int_v^\infty p(\theta, v, t) \phi_{y(v, t)} d\theta$$

The desired result now follows on observing that $J(v, t) \geq 0$ while $\gamma(v, t) \leq 0$ with equality holding only in those components for which the non-negativity constraint is not binding at v .

The interpretation of this result is clear. For all potential inputs at v , marginal revenue product is no greater than factor price with equality holding for actual inputs. What is perhaps a little surprising is that these magnitudes are all valued in the price system expected to hold at v , the date when the inputs are used.¹² It is natural that this

12. It is easy to see that the same result holds in all price systems when valuation is done using shadow prices.

should be so for factor price, but not quite so obvious that the resulting outputs should be valued as of the time of the input decision rather than as of the time they occur. Indeed, even the discount factor implicit in the prices cancels out, so that the result holds in undiscounted prices as well. Further, the money constraint plays no role at all -- the effects of the decision in producing later money do not matter here.

The reason for this is not hard to find. By using an input at v , the firm acquires a stream of outputs. An alternative way to acquire that stream is to buy it on the futures markets at v . The firm does not have to do either of these things, but if it does, they must be equivalent at the margin. It is the value of the outputs in the price system at v that is the opportunity cost of not using inputs then. Later prices and later money problems do not change that fact.

Since, in this model, the firm can arbitrage as well as produce, Theorem 3.1 does not exhaust the implications of the first-order conditions. One further implication is:

Theorem 3.2: Consider a pair of ordinary commodities, i and j , with respective dates θ_i and θ_j . Call these dated commodities (i, θ_i) and (j, θ_j) , respectively. Suppose that at some time v^* , $t \leq v^* < \text{Min}(\theta_i, \theta_j)$, the expected price ratio $p_i(\theta_i, v^*, t)/p_j(\theta_j, v^*, t)$ is decreasing in v . Then the short constraint for (i, θ_i) is binding at v^* .¹³

13. Here and elsewhere a constraint is considered binding if and only if its associated Lagrange multiplier is positive.

Proof: Differentiating the relevant parts of (3.10) with respect to v , we obtain:

$$\begin{aligned}
 (3.16) \quad & \frac{\partial \log p_i(\theta_i, v, t)}{\partial v} - \frac{\partial \log p_j(\theta_j, v, t)}{\partial v} \\
 &= \frac{\mu_j(\theta_j, v, t)}{\lambda_j(\theta_j, t) + I_j(\theta_j, v, t)} - \frac{\mu_i(\theta_i, v, t)}{\lambda_i(\theta_i, t) + I_i(\theta_i, v, t)}.
 \end{aligned}$$

However, the Lagrange multipliers and their integrals are all non-negative. Hence, if the left-hand side of (3.16) is negative at v^* , $\mu_i(\theta_i, v^*, t)$ must be strictly positive.

This result is the natural one that if the discounted price of (i, θ_i) is expected to fall relative to that of (j, θ_j) , then it will be optimal to arbitrage. Such arbitrage may not involve buying (j, θ_j) , because there may be a still more profitable opportunity, but it will certainly involve selling (i, θ_i) as short as possible.

Note that if the price ratio for the two goods is expected to remain constant, then the short constraints need not be binding, but they may be anyway if it is desirable to raise cash to take advantage of other opportunities. However, we can certainly state:

Corollary 3.1: If the firm plans to hold two commodities (i, θ_i) and (j, θ_j) over the entire interval $[v^*, v']$, $t \leq v^* < v' \leq \min(\theta_i, \theta_j)$, then $p_i(\theta_i, v^*, t)/p_j(\theta_j, v^*, t) = p_i(\theta_i, v', t)/p_j(\theta_j, v', t)$.

In particular,

Corollary 3.2: If at some time $v^* < \theta$, the firm plans to buy two θ -dated commodities, i and j , and to hold them until θ for use as inputs, then $p_i(\theta, v^*, t)/p_j(\theta, v^*, t) = p_i(\theta, \theta, t)/p_j(\theta, \theta, t)$.

Note that this does not imply that the price of each such input is expected to be the same at v^* as it is at θ , only that they are expected to change in the same proportion. The question of whether such prices are expected to change at all involves the money constraint, which so far has played no direct part. As one might expect,

Theorem 3.3: Consider a commodity, i , with date θ , and a time v^* , $t \leq v^* < \theta$.

(A) If, at v^* , $p_i(\theta, v, t)$ is decreasing in v , then the short constraint for the commodity is binding at v^* .

(B) If, at v^* , the money constraint is not binding but the short constraint for the commodity is binding, then $p_i(\theta, v, t)$ must be decreasing in v at v^* .

(C) If, at v^* , $p_i(\theta, v, t)$ is increasing in v , then the money constraint is binding at v^* .

(D) If, at v^* , the short constraint for the commodity is not binding, but the money constraint is binding, then $p_i(\theta, v, t)$ must be increasing in v at v^* .

(E) If, at v^* , neither the short constraint for the commodity nor the money constraint is binding, then $\partial p_i(\theta, v, t)/\partial v = 0$ at v^* .

(F) If both constraints are binding at v^* , then the sign of $\partial p_i(\theta, v, t)/\partial v$ at v^* is indeterminate.

Proof: Differentiate the i^{th} component of (3.10) with respect to v and rearrange terms, obtaining:

$$(3.17) \quad \frac{\partial p_i(\theta, v, t)}{\partial v} = \frac{p_i(\theta, v, t) \mu_0(v, t) - \mu_i(\theta, v, t)}{1 + J(v, t)}.$$

The desired results all follow immediately from the non-negativity of the Lagrange multipliers.

These results have a natural interpretation, as follows. Suppose first that the discounted price in question is expected to fall. It must then be optimal for the firm to sell as much as possible, being stopped only by the short constraint. On the other hand, it is possible for the firm to find it optimal to sell even if the discounted price is expected to rise, because it needs to raise money in order to invest in some other profitable opportunity to comply with Theorem 3.2 above or to purchase inputs. Hence, if the money constraint is binding, the short constraint can also be binding even with rising discounted prices. On the other hand, if the money constraint is not binding, then the short constraint can only be binding if discounted price is expected to fall.

If discounted price is expected to rise, then the undiscounted price will be rising faster than the discount rate. In this case, the firm will either be trying to buy the commodity or else will be investing in some other opportunity, so the money constraint must be binding. That constraint could be binding (because of other opportunities) even if discounted price is expected to fall, but that could only happen if the firm is simultaneously selling the commodity short to raise money. If the short constraint is not binding, then the money constraint can only be binding if discounted price is expected to rise.

Theorem 3.3 (which, like Theorem 3.2, will apply to households as well) has an interesting implication. Since it is always possible to transfer excess money forward in time at the instantaneous bond rate, the money constraint can only be binding at some $v' > t$ if it is equally binding at all v in the interval $[\underline{t}, v']$. This means that if there is any commodity whose discounted price is expected to rise at any time, v' , in the future, the money constraint will be binding up to that time. In that case, however, any commodity whose discounted price is not expected to rise before v' must have its short constraint always binding on any non-zero interval. This corresponds to the quite sensible behavior that, with a profitable investment opportunity at v' , goods whose dates occur no later than v' and in which there is no profit to be made from speculation will be purchased only when their dates come due if they are to be used as inputs. This will be done in order to transfer money forward to v' . When transaction costs are introduced, this conclusion must be appropriately modified and behavior will then be more continuous.

As the final result in this section, we can state:

Theorem 3.4: Suppose that the firm plans to buy or to hold at v a commodity i with date θ . Suppose that it plans to continue holding that commodity until at least v' ($t \leq v < v' \leq \theta$). Then

$$p_i(\theta, v, t)(1 + J(v, t)) = p_i(\theta, v', t)(1 + J(v', t)).$$

Proof: This follows immediately from (3.16) and the fact that the short constraint for the commodity is not binding at any time in $[\underline{v}, v']$.

Thus, in such a case, the rate of discounted price increase during the holding period must just offset the increase in the discounted shadow

price of money due to the purchase. This is a special case of the general proposition that at shadow prices marginal revenue product (in this case, the revenue from the sale evaluated using the shadow price of money at v' , $(1 + J(v', t))$), must just balance factor price (the cost of the purchase being valued using the shadow price of money at v , $(1 + J(v, t))$). These results are all very sensible.

4. Behavior of Households

In some respects, the analysis of the household parallels that of the firm, and, where it does, I shall use the same notation (Households will later be subscripted h , but the subscript will be omitted until it is required.). In one respect, however, the analysis of the household is more complicated than that of the firm; households, unlike firms, are permitted to hold shares of firms, and there is a market in such shares.

The household maximizes a strictly quasi-concave,¹ continuously

1. For convenience utility is defined over all goods including intermediate ones like pig-iron. It is to be understood that strict quasi-concavity only applies when the Fréchet derivative is not identically zero.

Fréchet differentiable utility functional, $U(c(\cdot, t), c_0(\cdot, t))$, where $c(\theta, t)$ is the n -vector of consumption of ordinary commodities which the household expects at time t to consume at time θ . $c_0(\theta, t)$ is the amount of money which the household expects at time t to consume at time θ .²

2. It is necessary to have money act as an ordinary commodity either in the production process or in consumption to avoid having its price go to zero in equilibrium. It is simplest to leave it out of production.

Obviously, this only makes sense for $\theta \geq t$. For $\theta \leq t$, $c(\theta, t)$ and $c_0(\theta, t)$ denote actual consumption (also denoted by $\bar{c}(\theta)$ and $\bar{c}_0(\theta)$ respectively).

Note that actual and planned consumption coincide at t . As with firms, I assume that the household is not merely planning to use promised but undelivered commodities at time t itself. In this model, disequilibrium at t consists of the inability to carry out purchase plans; disequilibrium resulting from default on contracts is assumed away.³ A

3. In large part this is a matter of not wishing to model the effects of default on the defaulter. Since the central feature of the model below is the assumption that surprises are never pleasant, it would probably be possible to accommodate the effects of default on the other party to the contract (although discontinuities in behavior would be awkward).

similar remark applies to the storage activities of the household, described below.

As with firms, the household has expectations as to discounted prices, denoted $p(\theta, v, t)$, as before. It also has purchase plans for ordinary commodities, again denoted by $r(\theta, v, t)$, with the same properties as for firms.

As already remarked, the household holds and trades shares in the firms. We denote by $k(v, t)$ the vector of holdings of shares which, at time t , the household plans to have at time $v \geq t$. Actual holdings are denoted by $\bar{k}(v)$. We normalize so as to make the total number of shares issued by any firm sum to unity and assume that all fractions between zero and one are possible (Short sales are discussed below.).

It is convenient to have a separate notation for the household's purchases of shares; we denote the vector of such purchases which, at t ,

the household expects to make at $v \geq t$ by $q(v, t)$. Purchases actually made are denoted by $\bar{q}(v)$. Obviously, such plans depend on prices; we denote the corresponding vector of discounted share prices which the household expects to encounter at v by $w(v, t)$. Actual discounted share prices at v are denoted by $W(v)$. (Note that shares are not distinguished by date.)

Now, the household, unlike the firm, can acquire money in more than one way. It can acquire (or spend) money through commodity and share trading and it can receive dividends. Postponing for the moment the treatment of dividends, denote the discounted amount of money which the household, at t , expects to acquire at $v \geq t$ through trading by $r_0(v, t)$ and actual such acquisitions by $\bar{r}_0(v)$. We can now write the first of the constraints on the behavior of the household by

$$(4.1) \quad \int_0^{\infty} p(\theta, v, t) r(\theta, v, t) d\theta + w(v, t) q(v, t) + r_0(v, t) = 0$$

for all $v \geq t$.

This simply states that the household must plan to exchange things of equal value when trading. (Recall that $r(\theta, v, t) = 0$ for $\theta < v$.)

Similar to the firm's production process, the household also has direct access to a process which turns earlier dated commodities into later dated ones. This process -- storage -- is given the same notation as was the firm's production process. Inputs into storage, planned at t , to be made at v are denoted by $y(v, t)$. The resulting planned availability of θ -dated goods is given by $\phi(y(\cdot, t), \theta, t)$, a Fréchet-differentiable functional.

All of the remarks made in the preceding section about the firm's production functional apply here. In addition, note that we can easily

accommodate the storage of perishable commodities (inputs yielding no later outputs) and endowments, such as labor, coming exogenously at different dates (outputs with no previous inputs) in the definition of $\phi(., ., .)$. It is convenient to assume that money (which is undated and thus automatically stored) is neither an input nor an output of the storage process.

Let $x(\theta, v, t)$ denote the stock of θ -dated goods which the household, at t , expects to have acquired through trade after the close of trade at $v \leq \theta \leq t$. For $v \leq t$, the actual stock on hand after trade at v will be denoted by $\bar{x}(\theta, v)$. Then:

$$(4.2) \quad x(\theta, v, t) \equiv \bar{x}(\theta, t) + \int_t^v r(\theta, a, t) da \quad v \geq t$$

and

$$(4.3) \quad \bar{x}(\theta, t) \equiv \bar{x}(\theta, 0) + \int_0^t \bar{r}(\theta, a) da,$$

where, as before, $\bar{r}(\theta, a)$ denotes actual purchases made at a .

Now, θ -dated commodities on hand after trade at θ can come either from purchase or from storage. Such commodities are either consumed or automatically become inputs into storage, since their date must be changed. We write this as:

$$(4.4) \quad x(\theta, \theta, t) + \phi(y(., t), \theta, t) - c(\theta, t) - y(\theta, t) = 0.$$

Combined with (4.2), this is a further constraint on household behavior.

We add the explicit non-negativity constraints that $c(\theta, t) \geq 0$, $c_0(\theta, t) \geq 0$, and $y(\theta, t) \geq 0$. As with firms, such technological constraints are not the only ones. In addition to (4.1) above, we add the requirement that the extent to which the household can plan to be short in θ -dated commodities approaches zero as θ gets closer. Thus, we assume the existence

of an n -component vector, $\varepsilon(\theta - v) \geq 0$, continuous in $(\theta - v)$ and with $\varepsilon(0) = 0$, such that

$$(4.5) \quad x(\theta, v, t) + \phi(y(\cdot, t), \theta, t) + \varepsilon(\theta - v) \geq 0$$

for all $\theta \geq v \geq t$.

I shall refer to these as "short constraints" for commodities.

There are also "short constraints" for securities. Households are allowed to sell shares short, but again not infinitely short. Moreover, while there is no need to require that short positions in securities be cleared by any particular time, since securities, unlike commodities, are undated, we still require that they be cleared eventually. Accordingly, we assume the existence of a non-negative vector $\kappa(v)$ with $\lim_{v \rightarrow \infty} \kappa(v) = 0$ such that

$$(4.6) \quad k(v, t) + \kappa(v) \geq 0$$

where

$$(4.7) \quad k(v, t) = \bar{k}(t) + \int_t^v q(a, t) da; \quad \bar{k}(t) = \bar{k}(0) + \int_0^t \bar{q}(a) da.$$

Now, as already indicated, the household expects to earn dividends on the shares it owns. Let $d(v, t)$ be a vector whose typical element is the total rate of dividend payments (discounted) that the household, at t , expects a particular firm to make at v (with actual payments denoted by $\bar{d}(v)$). Then the total discounted dividends which the household expects at t to receive at v can be written as $d(v, t) k(v, t)$.

Note that a household with a short position in the shares of some firm is obligated for the dividend payments accompanying such shares; this is the case in practice.

Like the firm, the household is subject to a "money constraint" requiring that its planned money stock never be negative. Denote the household's actual stock of money at t by $\bar{m}(t)$, then:

$$(4.8) \quad \bar{m}(t) = \bar{m}(0) + \int_0^t \bar{r}_0(a) da + \int_0^t \bar{d}(a) \bar{k}(a) da - \int_0^t \bar{c}_0(a) da$$

and the money constraints can be stated as:

$$(4.9a) \quad \bar{m}(t) + \int_t^v \{r_0(a, t) + d(a, t) k(a, t) - c_0(a, t)\} da \geq 0$$

for all $v \geq t$;

$$(4.9b) \quad \bar{m}(t) + \int_t^\infty \{r_0(a, t) + d(a, t) k(a, t) - c_0(a, t)\} da = 0.$$

As with firms, note that households take into account the fact that purchases must be made with money. Equation (4.9b) ensures that the household does not plan to hold a positive money stock forever -- an obviously harmless restriction. Without such a separate statement, one can encounter a slightly inconvenient anomaly in cases in which the money constraints are not binding at any finite time. In such a case, without (4.9b), the Lagrange multipliers for the money constraints (4.9a) would all be zero but would have a positive integral.

The Lagrangian for the household's optimization problem is given by:

$$(4.10) \quad L(t) = U(c(\cdot, t), c_0(\cdot, t)) \\ - \int_t^\infty \beta(v, t) \left\{ \int_0^\infty p(\theta, v, t) r(\theta, v, t) d\theta + w(v, t) q(v, t) + r_0(v, t) \right\} dv \\ - \int_t^\infty \lambda(\theta, t) \left\{ c(\theta, t) + y(\theta, t) - \phi(y(\cdot, t), \theta, t) - \bar{x}(\theta, t) - \int_t^\theta r(\theta, a, t) da \right\} d\theta$$

$$\begin{aligned}
& + \int_t^\infty \int_t^\theta \mu(\theta, v, t) \left\{ \bar{x}(\theta, t) + \int_t^v r(\theta, a, t) da + \phi(y(\cdot, t), \theta, t) + \varepsilon(\theta - v) \right\} dv d\theta \\
& + \int_t^\infty \mu_0(v, t) \left\{ \bar{m}(t) + \int_t^v (r_0(a, t) + d(a, t) k(a, t) - c_0(a, t)) da \right\} dv \\
& + \mu_0 \left\{ \bar{m}(t) + \int_t^\infty (r_0(a, t) + d(a, t) k(a, t) - c_0(a, t)) da \right\} \\
& + \int_t^\infty \alpha(v, t) \{k(v, t) - \kappa(v)\} dv + \int_t^\infty \gamma(v, t) y(v, t) dv + \int_t^\infty \delta(v, t) c(v, t) dv \\
& + \int_t^\infty \delta_0(v, t) c_0(v, t) dv + \int_t^\infty \eta(a, t) \left\{ \bar{k}(t) + \int_t^a q(v, t) dv - k(a, t) \right\} da,
\end{aligned}$$

where $\lambda(\cdot, t)$, $\mu(\cdot, \cdot, t)$, $\alpha(\cdot, t)$, $\gamma(\cdot, t)$, $\delta(\cdot, t)$, and $\eta(\cdot, t)$ are vectors of Lagrange multiplier functions, $\beta(\cdot, t)$, $\mu_0(\cdot, t)$, and $\delta_0(\cdot, t)$ are scalar Lagrange multiplier functions, and μ_0 is a Lagrange multiplier. I have deliberately adopted a notation paralleling that used for firms, using the same symbols of the Lagrange multiplier functions corresponding to similar constraints in the two cases.

The first-order conditions for a maximum are:⁴

4. Differentiation is in the following order: c , c_0 , y , r , r_0 , q , and k .

$$(4.11) \quad U_{c(\theta, t)}(c(\cdot, t), c_0(\cdot, t)) = \lambda(\theta, t) - \delta(\theta, t) \quad \theta \geq t$$

$$(4.12) \quad U_{c_0(\theta, t)}(c(\cdot, t), c_0(\cdot, t)) = \mu_0 + \int_\theta^\infty \mu_0(v, t) dv - \delta_0(\theta, t) \quad \theta \geq t \quad ^5$$

5. See the footnote after equation (3.11) above.

(4.13)

$$\lambda(v, t) - \gamma(v, t) = \int_v^\infty \left\{ \lambda(\theta, t) + \int_v^\theta \mu(\theta, a, t) da \right\} \phi_{y(v, t)}(y(\cdot, t), \theta, t) d\theta,$$

$$v \geq t,$$

$$(4.14) \quad \lambda(\theta, t) + \int_v^\theta \mu(\theta, a, t) da = \beta(v, t) p(\theta, v, t) \quad \theta \geq v \geq t,$$

$$(4.15) \quad \beta(v, t) = \mu_0 + \int_v^\infty \mu_0(a, t) da \quad v \geq t,$$

$$(4.16) \quad \beta(v, t) w(v, t) = \int_v^\infty \eta(a, t) da \quad v \geq t,$$

and

$$(4.18) \quad \eta(v, t) = d(v, t) \left\{ \mu_0 + \int_v^\infty \mu_0(a, t) da \right\} + \alpha(v, t) \quad v \geq t.$$

Despite their complex appearance, these conditions have quite natural interpretations (which are rather richer than those for the firm). I begin with the most familiar looking ones.

Theorem 4.1:

$$(4.19) \quad U_{c(\theta, t)} \leq \beta(\theta, t) p(\theta, \theta, t)$$

$$(4.20) \quad U_{c_0(\theta, t)} \leq \beta(\theta, t),$$

with the equalities holding for those commodities (including money) which the household actually plans to consume at θ .

Proof: Evaluate (4.14) at $v = \theta$ and substitute the result into (4.11).

Substitute (4.15) into (4.12). The desired result now follows on observing that $\delta(\theta, t) \geq 0$ and $\delta_0(\theta, t) \geq 0$, with equality holding for those commodities for which the non-negativity constraint on consumption is not binding.

This result is the expected one that marginal utility is proportional to discounted price for commodities actually consumed. Note that the discounted prices in question are those of the closing price system as of the date of consumption, which makes perfect sense.

Note further that, if money is actually consumed at θ , then $\beta(\theta, t)$ -- the Lagrange multiplier for the budget constraint and the factor of proportionality in the result -- works out to be the instantaneous marginal utility of money consumption at θ in the usual way. If money is not consumed at θ , this is not the case. The general interpretation of $\beta(\theta, t)$ is given by (4.15) where it can be seen to be the full marginal utility of money at θ taking account of the effect that a little more money at θ has on the availability of money at all later times. This leads immediately to:

Corollary 4.1: Consider two dates, v and v' , with $t \leq v < v'$. Suppose that at no time in the interval $[v, v')$ is the money constraint binding. Then for any good (including money) which the household plans to consume at any date in that interval, the instantaneous marginal utility of consumption is proportional to the discounted price obtaining at the instant of consumption, with the same factor of proportionality holding at all such dates.

Proof: This follows from Theorem 4.1, equation (4.15) and the fact that $\mu_0(a, t)$ is the Lagrange multiplier associated with the money constraint at a .

Thus, while marginal utility proportional to price can fail for commodities consumed at different times, it holds over any interval in which the money constraint is not binding.

Of course, as Theorem 4.1 makes clear, there is a sense in which marginal utility is always proportional to discounted price for consumed goods. In such an interpretation, the price of money is not really unity; in consuming money at θ , one must consider its opportunity cost in terms of the effects on later money constraints. That cost is measured by $\beta(\theta, t)$. Similarly, the opportunity cost of consuming an ordinary commodity at θ is not just given by its price; rather one must consider the possibility that refraining from such consumption and selling the commodity would have in alleviating later money shortages. The full such opportunity cost is measured by $\beta(\theta, t)$ times the price of the commodity in question. Viewed in such opportunity cost terms, marginal utility is always proportional to "price" for consumed goods.

A similar construct to $\beta(\theta, t)$ appeared in the analysis of the firm given above where it was denoted $J(\theta, t) + 1$.⁶ Indeed, using $\beta(v, t)$

6. The unit element in $J(v, t) + 1$ reflects the fact that even with all money constraints nonbinding for finite v , money is worth something to the firm. The same thing is reflected for the household by the appearance of μ_0 in $\beta(v, t)$. See (4.15).

in place of $(J(v, t) + 1)$ in the proof of Theorem 3.1 immediately yields:

Theorem 4.2:

$$(4.21) \quad \int_v^{\infty} p(\theta, v, t) \phi_{y(v,t)} d\theta \leq p(v, v, t),$$

with the equality holding for those commodities which the household actually plans to use as inputs to storage at v .

This proposition is exactly the same as that obtained for the production activities of firms. Indeed, it states that the household runs its storage activity exactly as if it were a profit-maximizing firm, valuing inputs and outputs as of the time when the input decision is taken.

(A heuristic explanation of why that is the right price system for such valuation was given in the discussion of firms, above.) This is not a surprising result. As with firms, the household believes it has a choice between putting something into storage and selling it on the open market; it also believes it has a choice between taking things out of storage and purchasing them on the open market. It is only rational for it to use such valuations in deciding whether or not to store.

Not surprisingly, the household also acts as does the firm in its decisions about speculation on the commodity markets. It is easy to see that Theorems 3.2, 3.3, and 3.4 apply directly to the household, since the first-order condition for the firm (3.10) is exactly the same as the first-order condition for the household (4.14) with $\beta(v, t)$ and $(J(v, t) + 1)$ playing the same role.⁷ Corollaries 3.1 and 3.2 also hold and the latter

7. Note that one can interpret the money constraint being binding at v^* as there being no change in the direct marginal utility of money at that time.

applies to goods held either for consumption or for storage inputs. I shall not restate all these results explicitly.

Note that Theorem 3.4 applied to households states that if the household plans to buy a commodity at v^* and resell it at v' with no intervening transactions, then, for that commodity, $\beta(v^*, t) p_1(\theta, v^*, t) = \beta(v', t) p_1(\theta, v', t)$. This can be interpreted in two ways. First,

as for the firm, the discounted marginal revenue from the sale must equal the discounted marginal cost of the purchase, with both being evaluated using the appropriate shadow prices reflecting the opportunity costs of money at the same dates. Alternatively, the marginal utility of the money invested in making the purchase must just be balanced by the marginal utility of the money received from the sale.

The speculative behavior of the household is richer than that of the firm, however, because the household is allowed to deal on the securities markets.

Theorem 4.3: (A) Suppose that the household plans to buy or to hold shares in the f th firm at some date $v \geq t$ and to hold such shares until some $v' > v$ when it plans to begin to sell them. Then

$$(4.22) \quad \beta(v, t) w_f(v, t) = \int_v^{v'} \beta(a, t) d_f(a, t) da + \beta(v', t) w_f(v', t).$$

(B) Moreover, it is always the case that

$$(4.23) \quad \beta(v, t) w(v, t) \geq \int_v^{\infty} \beta(a, t) d(a, t) da,$$

with equality holding for those firms whose shares the household plans to hold forever.

Proof: (A) From (4.16),

$$\begin{aligned} (4.24) \quad \beta(v, t) w(v, t) &= \int_v^{\infty} \eta(a, t) da = \int_v^{v'} \eta(a, t) da + \int_{v'}^{\infty} \eta(a, t) da \\ &= \int_v^{v'} \eta(a, t) da + \beta(v', t) w(v', t). \end{aligned}$$

Now substitute (4.15) into (4.18), evaluate the result at a , rather than v , and substitute it in the integral on the far right-hand side of (4.24), obtaining:

$$(4.25) \quad \beta(v, t) w(v, t) = \int_v^{v'} \{\beta(a, t) d(a, t) + \alpha(a, t)\} da + \beta(v', t) w(v', t).$$

The desired result now follows on observing that $\alpha(a, t)$ is the vector of Lagrange multipliers for the short constraints on securities at a and is zero in its f th component for all a in the interval $[\bar{v}, v')$.

(B) As before, substitute (4.15) into (4.18), evaluate the result at a and substitute it this time directly into (4.16). The desired result follows as before, since $\alpha(a, t) \geq 0$.

The result in (A) has the natural interpretation that when the household holds a stock over a period, the marginal utility of the money tied up in the stock at any moment of time must just balance the marginal utility of the dividend stream which the household expects to receive plus the marginal utility of the money it expects to receive from selling the stock. Similarly, (B) of the theorem states that the marginal utility of the money invested in the stock must be at least as great as the marginal utility of the corresponding expected dividend stream, with equality holding if the household plans to hold the stock indefinitely. Note that the marginal utility of money here is not the instantaneous marginal utility of money consumption (unless the household is actually consuming money at the relevant time) but the overall marginal utility of money taking account of the money constraint. The obvious alternative interpretation of the result in terms of the opportunity cost of money is left to the reader.

Parallel to Theorem 3.3 for speculation in commodities, however, we can state:

Theorem 4.4: Consider the household's investment in the shares of a particular firm, f , and a time, $v^* \geq t$. Define:

$$n_f(v, t) \equiv \frac{\partial w_f(v, t)}{\partial v} + d_f(v, t). \quad 8$$

8. Recall that the household subscript, h , continues to be omitted here. Thus, $d_f(a, t)$ is the f th component of $d(a, t)$. Later this will be written as $d_{hf}(a, t)$ and the entire vector as $d_h(a, t)$.

(A) If $n_f(v^*, t) < 0$, then the short constraint for shares of f is binding at v^* .

(B) If, at v^* , the money constraint is not binding, but the short constraint for shares of f is binding, then $n_f(v^*, t) < 0$.

(C) If $n_f(v^*, t) > 0$, then the money constraint is binding at v^* .

(D) If, at v^* , the short constraint for shares of f is not binding, but the money constraint is binding, then $n_f(v^*, t) > 0$.

(E) If, at v^* , neither the short constraint for shares of f nor the money constraint is binding, then $n_f(v^*, t) = 0$.

(F) If both constraints are binding at v^* , then the sign of $n_f(v^*, t)$ is indeterminate.

Proof: Differentiate (4.25) with respect to v and rearrange terms, obtaining:

$$(4.26) \quad n_f(v, t) = \frac{w_f(v, t) \mu_0(v, t) - \alpha_f(v, t)}{\beta(v, t)}. \quad 9$$

9. Note the parallel to (3.17).

The desired results now follow from the nonnegativity of the Lagrange multipliers.

These results are most easily interpreted by considering what it means for n_f to be zero. Ownership of a share of stock is ownership of a future dividend stream. Hence, the natural zero for the rate of change of the stock price is for that price to decrease at precisely the rate at which dividends are paid out. Indeed, the instantaneous rate of return on stock held at v is not the rate of price increase alone, but rather $n_f(v, t)/w_f(v, t)$.

This magnitude also turns out to be the natural one for the parallel to Theorem 3.2 when shares are involved, which is:

Theorem 4.5: Consider the shares of two firms, f and f' , a commodity, i , with date θ , and a time v^* , $t \leq v^* < \theta$.

(A) If, at $v = v^*$, $\partial \log p_i(\theta, v, t)/\partial v < n_f(v^*, t)/w_f(v^*, t)$, then the short constraint for (i, θ) is binding at v^* .

(B) If, at $v = v^*$, $\partial \log p_i(\theta, v, t)/\partial v > n_f(v^*, t)/w_f(v^*, t)$, then the short constraint for shares of f is binding at v^* .

(C) If $n_f(v^*, t)/w_f(v^*, t) < n_{f'}(v^*, t)/w_{f'}(v^*, t)$, then the short constraint for shares of f is binding at v^* .

Proof: These results all follow from (4.26) and (3.17) with $\beta(v, t)$ replacing $(1 + J(v, t))$. The details are left to the reader.

One final remark about the stock market before moving on: Differing expectations as to the profits of firms and differing expectations as to

prices make it natural to have a market for shares in the model. As earlier remarked, however, examination of the results shows that what really matters is not differing expectations or even uncertainty, but the fact that discounted prices are expected to change. Even if there is perfect foresight, the fact that the marginal utility of money, $\beta(v, t)$, will generally change with v leads to a useful role for a securities market -- the role of exchanging present for future money at rates which may differ from the rates involved in the bond market.

5. Transaction Costs and Transaction Constraints

So far we have proceeded as though agents always expect to be able to complete their planned transactions with no difficulty. This is hardly a reasonable assumption to make in a disequilibrium world where agents constantly experience transaction difficulties. There are (at least) two ways to incorporate such transaction problems into the analysis. One way is to introduce some form of transaction costs which depend smoothly on the size of the transaction; the other is to assume that agents believe themselves limited in the volume of transactions they can complete.

The latter case (which can, of course, be considered as one in which transaction costs become infinite at certain transaction volumes) has received much recent attention in the literature under the name of "quantity constraints."¹ I shall refer to it as the case of "transaction constraints"

1. See, for example, Barro and Grossman [2], Benassy [3, 4, 5, 6], Clower [8], Fisher [17], Frevert [18], Hahn [19], Leijonhufvud [24, 25], Patinkin [28], Varian [33, 34] and Veendorp [35]. Drazen [9] presents a survey.

to avoid any danger of confusing the constraints with the "short constraints" already introduced. Note that the transaction constraint and the smooth transaction costs treatments are not mutually exclusive; one can have smooth transaction costs up to the point at which the constraint is hit. The case with just transaction constraints simply has the costs zero up to that point.

In this section, I discuss the issues involved in both treatments. Later sections will not make such treatment explicit except where absolutely

necessary. As will eventually emerge, dealing with these problems does not alter the basic stability results in any very significant way (although the assumptions have to be slightly strengthened to deal with transaction constraints) except for the obvious and important fact that allowing transaction constraints admits the possibility that the equilibrium to which the system tends is not a Walrasian equilibrium.

I begin, then, with the case of smooth transaction costs. Since it is unwieldy to deal with an explicit transactions technology, I assume that transactions volumes enter explicitly into the utility functionals of households and into the production functionals of firms. The notion here is that agents expect to have to use up resources in completing transactions (at least if they are on the wrong side of a particular market); this results in a diminution of utility for the household and in less output for given input for the firm than would otherwise be obtained. One can think of the resources involved as those which have to be expended to find someone who is willing to trade at the going prices. It is fairly natural to assume that such efforts are continuous and the marginal effort required increasing in the size of the transaction.²

2. Such continuity is not as innocuous as it appears. It rules out the case in which the efficient way to accomplish any non-zero transaction (no matter how small) is to set up a shop or take out a fixed cost piece of advertising. The increasing nature of marginal transaction costs appears below as an assumption of strict quasi-concavity.

We may think of smooth transaction costs in at least two other ways as well. The first of these, related to what has just been said,

is in lieu of a price premium paid to induce others to transact when transactions are difficult. The difference, of course, lies in the fact that in the treatment given here, the premium is paid, as it were, but it is not received. I shall have a bit more to say about this below.

The second way is as a rather ad hoc way of introducing uncertainty which is otherwise conspicuously absent from the model. Perhaps the most undesirable feature stemming from that absence is the bang-bang nature of the solutions to agents' optimizing problems. As prices change, agents undertake large changes in behavior in pursuit of vanishingly small profits because they feel sure of getting those profits. Introducing transaction costs can ensure that agents change their behavior substantially only if they expect substantial rewards for doing so. In this sense, transaction costs can be thought of as risk premiums which need to be paid.

It is also a matter of considerable technical convenience to be able to assume that the optimizing behavior of agents is unique and continuous in the prices. As just indicated, one can do so with smooth transaction costs and I shall. The appropriate assumption turns out to be that optimum consumption, production, purchase, and storage plans are all unique and continuous and Lipschitzian in past history and the prices.³

3. Assuming continuity rather than upper semi-continuity in the prices (and hence strict quasi-concavity rather than just quasi-concavity in transaction costs) is largely a matter of convenience. Mostly what is needed is the existence of a solution to the differential equations of the system (and, for some purposes, the continuity of that solution in the initial conditions). See Champsaur, Drèze, and Henry [7]. Note that, at least with transaction costs added (and to some extent without

them) constant returns technologies can be accommodated, as was not the case in earlier papers. See Fisher [13, 14, 16] and Arrow and Hahn [1]. A strengthening of the uniqueness assumption is required below to ensure convergence to a point rather than to a set of equally optimal programs. Finally, note that the stronger assumption of weak* continuity is not required. Fisher [16] requires it unnecessarily. I shall return to these matters below.

With this in mind, let us turn to the formal set-up. I deal explicitly only with households, since firms present no different problems and are in fact a little simpler since no proof of strict quasi-concavity for the parallel of the concentrated utility functional introduced below is required. In a notation restricted entirely to this section, expand the arguments of the household's utility functional so that it becomes $U(c(\cdot, t), c_0(\cdot, t), r(\cdot, \cdot, t), q(\cdot, t))$. I shall assume that this is strictly quasi-concave, and Fréchet differentiable in all arguments. As regards the Fréchet derivatives with respect to transactions, I shall assume:

$$(5.1)^4 \quad U_{r_i}(\theta, v, t) r_i(\theta, v, t) \leq 0; \quad U_{q_i}(v, t) q_i(v, t) \leq 0.$$

4. See the footnote after equation (3.11) above.

It is easy to see what the introduction of transaction costs in this way does to the optimizing behavior of the household. Basically, transaction costs get added to the costs of buying and selling and must now be taken into account. More specifically, the first-order conditions which are affected are (4.14) and (4.16). The former equation has a term of

$U_{r(\theta, v, t)}$ subtracted from its right-hand side; the latter one has a term of $U_{q(v, t)}$ subtracted from its left-hand side. Define:

$$(5.2) \quad p'(\theta, v, t) \equiv p(\theta, v, t) - \frac{U_{r(\theta, v, t)}}{\beta(v, t)}$$

$$(5.3) \quad q'(v, t) \equiv q(v, t) - \frac{U_{q(v, t)}}{\beta(v, t)},$$

so that $p'(\theta, v, t)$ and $q'(v, t)$ are full discounted "prices" including the monetary equivalent of the marginal transaction disutilities. All of the theorems of the preceding section now hold with these "virtual prices" replacing the actual ones.

I shall not bother to restate all those results explicitly, but shall comment briefly on some of the interpretations. In the first place, it is not now true that marginal utility is proportional to discounted money prices (as opposed to the virtual prices just defined) for commodities planned to be consumed at a date θ , unless such planned consumption is to take place entirely out of stocks acquired before θ . To put it another way, in view of (5.1), consumption can take place with marginal utility more than proportional to discounted money price if it would be necessary to acquire the good at the last moment in order to make proportionality hold; similarly, consumption can take place with marginal utility less than proportional to discounted money price if it is necessary to sell the good to achieve proportionality. Note that transaction costs before θ play no direct role in this (they play an indirect one because it will generally be the case that optimal transactions are continuous in the transactions date); by the time θ comes around, they are bygones in valuing consumption.

Transaction costs of many different dates do enter into the extension of Theorem 4.2 to the present case, however (as in the parallel result for firms). Here the household must not only value the cost of a marginal storage input to include the transaction costs (if any) associated with acquiring it but must also value the marginal stream of outputs that it produces taking into account the transaction costs which it would have to occur if it were to acquire such outputs through purchase at the date of the input. This fits nicely with the discussion in previous sections as to why it is the price system holding at the input date which occurs here.

The speculative results (Theorems 3.2, 3.3, and 3.4 applied to either firms or households and Theorems 4.3 and 4.4) also carry over naturally. Essentially, the household only engages in speculation if the virtual prices defined above promise a profit, that is, if the monetary profit involved is sufficient to cover the transaction costs. This comes across most clearly in the theorems as to speculative holding of commodities and shares (Theorems 3.4 and 4.3, respectively) where the result becomes precisely that the marginal utility of the money received when the item is sold plus the marginal utility of the dividends received while holding it in the case of shares less the marginal utility of the money tied up in it at the time of purchase must just balance the sum of the marginal utilities lost through the transaction costs of purchase and sale. The exact formal statement is left to the reader. Note that "speculative" purchases or sales may occur with zero monetary price change if the difficulty of later transactions is expected to be different from that of earlier ones.

Similarly, the discontinuous behavior discussed after Theorem 3.3 above becomes continuous when smooth transaction costs are introduced.

While it remains true that if the money constraint is binding at $v' > t$, then it is equally binding at all times in the interval $[\underline{t}, v']$, it is no longer true that all goods whose prices do not rise before v' will be bought (if at all) for input or consumption purposes only as their dates come due. This would be so only if the discounted money prices of such goods dropped sharply enough up to those dates to make their virtual prices non-increasing when purchases are massed at the last possible moment (and this may be impossible). Goods whose discounted money prices drop less sharply than this will begin to be bought continuously as their dates approach.

Before moving on to the case of transaction constraints, one further matter requires discussion. I shall show below that the economy converges to an equilibrium supported by a price system. Without transaction constraints that equilibrium will be a Walrasian one; even with transaction constraints the equilibrium will be utility maximizing for households and profit maximizing for firms within the constraints at the equilibrium prices. In either case it will be necessary to show that smooth transaction costs of the type under discussion make no difference to this property in equilibrium (although they will certainly affect the nature of the equilibrium set). I now discuss the point involved explicitly for households, the discussion for firms being similar.

An equilibrium will turn out to be a point at which discounted prices⁵ are constant and are expected to be so by all agents for the

5. These are money prices. When virtual prices (defined in (5.2) and (5.3)) recur in the discussion, they will be so labelled.

commodities or shares in which they plan to trade. Consider the household planning in equilibrium. In considering alternative consumption programs, the household will find that the discounted cost of achieving any such program is independent of the transactions path used to get to it. Hence, the transactions path used to achieve any consumption program will be chosen solely to maximize the utility functional given $c(., t)$ and $c_0(., t)$. Moreover, such optimizing transaction paths will depend only on $c(., t)$ and $c_0(., t)$ and not on the value of equilibrium prices.

Thus, for any set of equilibrium prices, let

$$(5.4) \quad r(., ., t) = F(c(., t), c_0(., t)); q(., t) = G(c(., t), c_0(., t))$$

as the utility maximizing transaction paths. Define the "concentrated utility functional", $V(c(., t), c_0(., t))$ as the functional obtained when (5.4) is substituted into $U(., ., ., .)$. It is obvious that (still at equilibrium), the household acts to maximize $V(., .)$ given the prices (and the various constraints).

It now almost follows that in examining equilibria, we can consider the concentrated utility functional, $V(., .)$ as an ordinary utility functional without transaction costs. To do this, however, we need to show that $V(., .)$ is strictly quasi-concave.

Lemma 5.1: The concentrated utility functional, $V(., .)$ is strictly quasi-concave.

Proof: Choose any scalar a with $0 < a < 1$ and let $\{c(., t), c_0(., t)\}$ and $\{c'(., t), c'_0(., t)\}$ be any two different consumption programs with $V(c, c_0) = V(c', c'_0)$.⁶ Let $c'' = ac + (1-a)c'$ and $c''_0 = ac_0 + (1-a)c'_0$.

6. I omit the arguments of expressions like $c(., t)$ where they are merely burdensome.

Let $r = F(c, c_0)$, $r' = F(c', c'_0)$, $r'' = F(c'', c''_0)$, and similarly for q , q' , and q'' . Then:

$$(5.5) \quad V(c'', c''_0) = U(c'', c''_0, r'', q'') \geq U(c'', c''_0, ar + (1-a)r', aq + (1-a)q') \\ > U(c, c_0, r, q) = V(c, c_0),$$

where the first inequality follows because (r'', q'') is the utility-maximizing transactions path with which to achieve (c'', c''_0) , while that consumption program can certainly be achieved by using the transactions path given by $(ar + (1-a)r', aq + (1-a)q')$. The second inequality follows because $U(c, c_0, r, q)$ is strictly quasi-concave.

I now drop the explicit transactions arguments from the utility functional although I shall interpret later results to include them and shall assume them present, so that continuity in the prices is maintained as already discussed.

This brings us to the discussion of transaction constraints.

It is simplest to begin with the formal development. Suppose that at time t , the household (as before, firms are similar) planning its transactions in θ -dated goods at v believes that it is constrained by:

$$(5.6) \quad -s(\theta, v, t) \leq r(\theta, v, t) \leq b(\theta, v, t),$$

where $s(\theta, v, t) \geq 0$ and $b(\theta, v, t) \geq 0$ are out of the agent's control.

These constraints add two new terms to the Lagrangian (4.10), namely,

$$(5.7) \quad \int_t^\infty \int_t^\theta \zeta(\theta, v, t) \{r(\theta, v, t) + s(\theta, v, t)\} dv d\theta \\ + \int_t^\infty \int_t^\theta \xi(\theta, v, t) \{b(\theta, v, t) - r(\theta, v, t)\} dv d\theta.$$

Obviously, at most one of $\zeta_1(\theta, v, t)$ and $\xi_1(\theta, v, t)$ will be nonzero.

Similarly, suppose that there are constraints on transactions in shares:

$$(5.8) \quad -S(v, t) \leq q(v, t) \leq B(v, t)$$

with corresponding terms in the household's Lagrangian:

$$(5.9) \quad \int_t^\infty \tau(v, t) \{q(v, t) + S(v, t)\} dv + \int_t^\infty \psi(v, t) \{B(v, t) - q(v, t)\} dv.$$

Only one of $\tau_f(v, t)$ and $\psi_f(v, t)$ will be nonzero.

Now consider the first-order conditions (4.11) - (4.18). The only ones affected (as in the case of smooth transaction costs, which may also be present) will be (4.14) and (4.16). The former equation will have $\{\xi(\theta, v, t) - \zeta(\theta, v, t)\}$ added to its right-hand side; the latter equation will have $\{\psi(v, t) - \tau(v, t)\}$ added to its left-hand side. Not surprisingly, these terms play the same respective roles as $-U_{r(\theta, v, t)}$ and $-U_{q(v, t)}$ did in the discussion of smooth transaction costs above (Note that the latter terms may also be present.); they are simply terms showing the marginal effect of transactions on utility. The remainder of the analysis of the effects on individual behavior in terms of virtual prices, as in (5.2) and (5.3), will not be repeated. (Note that the sign restrictions of (5.1) will be automatically obeyed by the new terms.)⁷

7. There is one complication here which it seems best to avoid.

The theory of discounting explored in Section 2, above, assumed that agents discounted the future because they could always deal on the bond market, in particular in bonds of very short maturity. If there are transaction constraints on dealing in such bonds, and, indeed, even if there are transaction costs to such transactions, then the instantaneous discount rate for each agent will not depend only on the bond prices but will reflect the shadow prices of the constraints and the disutilities of the transaction. Since these will generally depend on the size of the bond market transaction, the opportunity costs of holding money will be nonlinear in the amount of money held. This makes the optimization problems of individual agents quite messy, although essentially the same as before at the margin with appropriate virtual prices including a virtual discount rate. Moreover, it makes the statement of equilibrium conditions below rather complicated to have to work with virtual rather than actual bond prices. Accordingly, I shall simplify matters and assume that every agent discounts the future at an instantaneous discount rate given by the profile of current bond prices; in effect, this means assuming that transaction costs or constraints associated with dealings in bonds go to zero with the maturity of the bond.

The close resemblance between the effects of smooth transaction costs and transaction constraints ends here, however. In particular, whereas we were able to incorporate smooth transaction costs into a "concentrated utility functional" (or "production functional") and deal with equilibrium points as Walrasian after so doing, this cannot be done

with transaction constraints. Indeed, the possibility of non-Walrasian equilibria is a very real one. Although, as we shall see below, the proof of stability remains unaffected (except for some relatively minor additions which must be made to the assumptions), there is now no guarantee that the equilibrium to which the system converges will be a Walrasian one.⁸

8. That such non-Walrasian equilibria exist when there are transaction constraints has been demonstrated extensively in the literature. See Hahn [19] for a recent example. The stability properties are less well explored, however. (See Veendorp [34].)

The present analysis, so far as I know, is the first to do so in a context of full optimization on the part of agents in the presence of transaction constraints. In this respect, it differs from the rather less satisfactory treatment given in Fisher [17] where the use of certain rules-of-thumb by agents in a two-stage utility maximization process leads to stability of Walrasian equilibrium under appropriate assumptions.

This point is rather closely related to one made in the introduction as to perfect foresight. If all agents perceive their transaction constraints correctly (or perceive themselves as more restricted than in fact they are), then stability is trivial since all planned transactions will be completed and the system will not move. It is the mistaken belief that transactions can be completed which keeps things moving out of equilibrium. The possibility of convergence to non-Walrasian equilibrium occurs because, as the system moves and agents revise their perceptions of the transaction constraints, it can perfectly well happen that their perceived constraints become correct (or too tight) before a Walrasian

equilibrium is reached. It may be silly to suppose that such a situation obtains at every point (perfect foresight as to constraints), but it is certainly not silly to suppose that it can happen eventually.

One can dig a little deeper than this, however, and ask why constraints such as (5.6) and (5.8) are perceived in that form at all. As those constraints are formulated, agents believe that such restrictions do not depend on prices. In particular, they are not allowed to experiment and discover that in fact they can loosen the constraints by making better price offers. This is not very sensible. What can be said in extenuation, and what difference is it likely to make?

In the first place, it is not the case that there is no relation between price changes and transaction constraints in this (or similar) models. As will be described in a later section, prices move in the direction of excess demands in essentially the traditional way. If markets are orderly (binding constraints only on one side of the market), then an agent encountering a transaction constraint can reasonably expect prices in that market to change in such a way as to weaken the constraint encountered by requiring him to pay a premium. Nevertheless, individual agents are kept from making price offers to get around transaction constraints, and prices are set impersonally.

The matter is deeper than this, however. As shown in Fisher [11], we can think of identifying commodities by the name of the seller (or buyer, depending on the institutional arrangements) and having each dealer responsible for setting his own price. Such a treatment leaves stability results such as that obtained below essentially unchanged and has a number of attractive properties.⁹ In such an interpretation, the price-setting

9. This interpretation will be kept in the background in the present paper but used to justify the absence of the dated commodities problem, as discussed in the introduction.

agents do indeed react to transaction constraints in a certain sense. If they turn out to be mistaken about such constraints, believing, for example, that they can sell more than they actually do, then they will find their wares in excess supply and will lower the price. It is also true that if they are mistaken in a different sense and find that there is more demand than anticipated, then they find their wares in excess demand and will raise the price. In effect, the latter situation comes about when the other side of their personalized markets errs as regards transaction constraints.

The crucial thing, therefore, is not that agents do not change prices in order to alter transaction constraints; rather it is that they do not do so optimally.¹⁰ Prices are only adjusted after mistakes are made and

10. See Rothschild [29].

no account is taken in setting the price of the way in which the constraints will vary with price. To put it differently, out of equilibrium, each seller has a little monopoly power. In the treatment just described, he does not use it, for it is not generally true that moving prices in the direction of excess demand is the optimal thing for a monopolist to do.

In my opinion, the lack of rational price-setting out of equilibrium is (with the possible exception of the non-treatment of uncertainty) the most glaring defect in the present model, transaction constraints

or no transaction constraints. But the subject is very hard. We lack any very rigorous theory of how prices in competitive markets change out of equilibrium through the actions of rational agents, and the present paper just does not really deal with that subject.¹¹

11. For an attempt at the level of the individual market, see Fisher [12].

In one sense, however, we have already introduced a not very satisfactory substitute for allowing agents to take account of the way in which transaction constraints vary with the offer of price premia. As already discussed, one way of thinking of smooth transaction costs is as representing the premia which must be paid to find someone with whom to transact. The catch is that while such premia are paid in terms of reduced utility or output by the person seeking to promote the transaction (and these can be thought of in monetary equivalents as in (5.2) and (5.3)), they are not received by the other. In this crucial respect, fictitious "prices" and actual prices are not the same.

That this makes a big difference is already evident. If we dealt with the problem only through the introduction of smooth transaction costs as above, then, as we have seen, the analysis can remain in essentially Walrasian terms. If attempting to overcome transaction constraints requires adjustments to actual, rather than fictitious prices, however, then, as Hahn [19] has shown, we have to expect non-Walrasian equilibria to continue to occur.¹² What difference allowing price premia to be

12. As Maskin has remarked, however, Hahn's rather remarkable result

may depend on the assumption that the only price changes contemplated are by agents whose constraints are binding. In this respect, the dealers in the individualistic version of the present paper discussed above are smarter than the otherwise more rational agents in Hahn, for they change prices also when the other sides of their markets hit transaction constraints.

paid would make out of equilibrium is an important matter for further work.¹³

13. In this connection, it might be remarked that the agents in Hahn's model never in fact pay such premia but only contemplate them, since Hahn's analysis is directed at establishing the existence of equilibria where agents are content to live with their transaction constraints.

6. Closure Equations, Walras' Law, and Dividends

So far, I have dealt with the behavior of individual agents without much regard for their interaction. We must now begin to consider the equations which link their behavior.

The first such equations are readily stated. Recalling that actual purchases made at $v \leq t$ are denoted by $\bar{r}(\theta, v)$ and $\bar{q}(v)$ and introducing the subscripts h for households and f for firms, the fact that the economy is closed implies:

$$(6.1) \quad \sum_f \bar{r}_f(\theta, v) + \sum_h \bar{r}_h(\theta, v) = 0 \quad \theta \geq v \leq t$$

and

$$(6.2) \quad \sum_h \bar{q}_h(v) = 0 \quad v \leq t.$$

Further, since money is not produced (or used up) and since the dividends actually paid by firms must equal the dividends actually received by households,

$$(6.3) \quad \sum_h \bar{r}_{0h}(v) = - \sum_f \dot{\pi}_f(v) \equiv \sum_f \int_0^\infty P(\theta, v) \bar{r}_f(\theta, v) d\theta.$$

The derivation and nature of Walras' Law for this economy is more complicated than this, as can be seen by realizing that future demands of agents are valued at their own expected prices. We indicate those expectations by subscripting the prices appropriately.

We begin with households. Planning at t , the household's demand for money inflow at $v \geq t$ consists of its planned consumption of money at v plus the planned increase in its money stock at v . That money stock is given on the left-hand side of (4.9a) and we see that the household's demand for money inflow at v (not surprisingly) consists of the

money it expects to take in through trade plus the dividends it expects to receive. Using the budget constraint (4.1), we obtain:

$$(6.4) \quad \{r_{0h}(v, t) + d_h(v, t) k_h(v, t)\} + \int_v^{\infty} p_h(\theta, v, t) r_h(\theta, v, t) d\theta \\ + w_h(v, t) q_h(v, t) = d_h(v, t) k_h(v, t)$$

which can be read as saying that the household's demand for money inflow at v plus the value of its demand for inflow of ordinary commodities at v plus the value of its demand for inflow of shares at v must equal the value of the dividends which the household expects to take in at v , where all values are in the price system which the household expects to encounter at v .

Now consider firms. Let $g_f(v, t)$ be the rate at which firm f , planning at t , expects to pay dividends at $v \geq t$. This can be written as the expected money inflow through trade less the expected change in retained earnings at v . This amounts to:

$$(6.5) \quad \{- \int_v^{\infty} p_f(\theta, v, t) r_f(\theta, v, t) d\theta - g_f(v, t)\} \\ + \int_v^{\infty} p_f(\theta, v, t) r_f(\theta, v, t) d\theta = -g_f(v, t)$$

which can be read as saying that the firm's demand for money inflow at v plus the value (at its own expected prices) of its demand for inflow of ordinary commodities at v must be the negative of the dividends it expects to pay at v .

Now, sum (6.4) over households and (6.5) over firms and add the result. We obtain a generalized version of Walras' Law which states:

Theorem 6.1: For every $v \geq t$, value each agent's demands (the ones which will be current at v) at the prices which that agent expects to encounter at v . The total over all agents of the value of excess demands for inflows of money, ordinary commodities, and shares equals the difference between the total amount of dividends which households expect to receive at v and the total amount of dividends which firms expect to pay at v .

Since this holds for every $v \geq t$, a similar statement will hold for the total value of all excess demands integrated from t to infinity.

Thus, disparate expectations as to prices and dividends show up in this sort of alteration of Walras' Law for $v > t$. For $v = t$, however, since we assume all agents know correctly what is actually happening, such expectations are in fact all the same and we obtain the usual result:

Corollary 6.1: At t , valued in the actual prices, the total over all agents of the value of current excess demands for inflows of money, ordinary commodities, and shares is zero. That is:

$$(6.6) \quad R_0(t, t) + \int_t^{\infty} P(\theta, t) R(\theta, t, t) d\theta + W(t) Q(t, t) = 0$$

where

$$(6.7) \quad R_0(v, t) \equiv \sum_h r_{0h}(v, t) + \sum_f \left(- \int_v^{\infty} p_f(\theta, v, t) r_f(\theta, v, t) d\theta \right);$$

$$R(\theta, v, t) \equiv \sum_h r_h(\theta, v, t) + \sum_f r_f(\theta, v, t);$$

$$Q(v, t) \equiv \sum_h q_h(v, t).$$

Note that both Theorem 6.1 and Corollary 6.1 hold whether or not there are transaction constraints.

Now, as will be developed more formally below, equilibrium will require not only that all demands be fulfillable at given prices but also that all price expectations relevant to action be the same over agents and be actually fulfilled. It is clear from Theorem 6.1, however, that we shall also have to require that dividend plans are also fulfilled so that shareholding households and firms agree (and are correct) on the rate at which future dividends will be paid.¹ This requires a discussion

1. Note, however, that unlike Fisher [13, 14, 16] this will not require (even given (6.8) below) that all profits actually be paid out by the time equilibrium is reached. As with trades, what is required is that expectations be fulfilled, not that activity stop.

of the way in which dividend plans are set. For households this is largely a part of the discussion of No Optimistic Surprise given below, because it is largely a matter of expectation formation. For firms, however, it is a matter of policy.

I am not going to be very explicit about the way in which firms form their dividend policies. For purposes of the present analysis, it will suffice if four rather general conditions are met. First, dividend payments have to move smoothly enough in the other variables to permit the existence and continuity of solutions to the differential equations of motion of the system. Second, as already implicitly assumed, firms first plan so as to maximize profits and only then formulate dividend plans; they do not change their profit-maximizing activities to achieve dividend plans. Third, if a firm finds that its expectations are fulfilled as regards the prices of commodities it wishes to buy and sell, if it is in fact able to complete its attempted transactions and does

not alter its expectations as to future transaction constraints (or costs), then it does not change its dividend plans. Finally, since firms exist for the benefit of their stockholders, they expect asymptotically to pay out all their profits as time goes to infinity, that is (once more omitting subscripts):

$$(6.8) \quad \bar{G}(t) + \int_t^v g(v, t) dv = \pi(t). \quad 2$$

2. This is a sensible assumption but is not strictly required below. Without it, however, we would have the possibility that it not only is not true that all profits are paid out by the time equilibrium is reached (see the preceding footnote) but it also is not true that firms and households expect them ever to be completely paid out, even asymptotically, although all agents agree as to future dividend payments.

Within these limits, the dividend plans of the firm are a matter of negotiation between the firm and its stockholders. Both firms and households require liquidity for transaction purposes, but both have access to the bond market, and households can and will (see Theorem 4.3) adjust their holdings if they find that the dividend profiles expected from the firms in which they own shares no longer match their liquidity needs.

7. Equilibrium

Definition 7.1: An equilibrium is a state of the system and its history at which (7.1) - (7.6) below hold for all h , all f , all commodities, i , all θ , and all v with $\theta \geq v \geq t$:

$$(7.1) \quad R(\theta, v, t) \leq 0;$$

$$(7.2) \quad R_0(v, t) = 0;$$

$$(7.3) \quad Q(v, t) \leq 0;$$

$$(7.4) \quad \text{If } r_{hi}(\theta, v, t) \neq 0, \text{ then } p_{hi}(\theta, v, t) = P_i(\theta, t);$$

$$\text{If } r_{fi}(\theta, v, t) \neq 0, \text{ then } p_{fi}(\theta, v, t) = P_i(\theta, t);$$

$$(7.5) \quad \text{If } q_{hf}(v, t) \neq 0, \text{ then } w_{hf}(v, t) = W_f(t) - \int_t^v d_{hf}(a, t) da;$$

$$(7.6) \quad \text{If } k_{hf}(v, t) \neq 0, \text{ then } d_{hf}(v, t) = g_f(v, t).^1$$

1. $d_{hf}(v, t)$ is the f th component of $d_h(v, t)$; it gives the total dividends which (as of t) the h th household expects the f th firm to pay at v . A similar statement applies to other double subscripts noting that the subscripts for items relating to the household's dealings in shares are naturally those of the firms whose shares are involved.

A Walrasian equilibrium is an equilibrium in which none of the perceived transaction constraints, (5.6) and (5.8), is binding except possibly for constraints on the disposition of free goods.

Some comments are in order. First, unlike the case in earlier papers, there is no requirement that trade cease at equilibrium. Instead, what occurs is that trading plans are jointly feasible at the foreseen prices.

Hence what is required is that total excess demands be non-positive, not that individual excess demands be so. This raises the question of what ensures that specific individuals will be able to carry out their plans even if totals are feasible.

The answer here is that we shall be assuming a Hahn Process world²

2. We shall not do so explicitly, but, as discussed below, other assumptions, in particular that of No Optimistic Surprise only make sense in such a world. For the Hahn Process, see Hahn and Negishi [20].

in which, after trade, there is unsatisfied excess demand only on one side of any particular market.³ This makes it natural to assume (although

3. Indeed, if definable groups of agents were systematically separated as to their ability to trade with each other, it would make sense to redefine markets so as to consider them in separate markets. In Fisher [11], where commodities are distinguished by the identity of the dealer who sells them, it is observed that the Hahn Process assumption is almost compelled.

this is not necessary) that non-Walrasian equilibria are orderly in that perceived transaction constraints are binding at most on one side of each market.

Next, even though trade does not necessarily cease at equilibrium, the price expectations of those wishing to trade at some future date are that the discounted price for that trade will be the same as the current price. They are thus indifferent (so far as price is concerned) as to

the date of the transaction. There are two aspects of this that require discussion: the fact that it is discounted prices which are expected to be unchanged and the fact that such expectations only apply to those desiring to trade.

It is plain from a consideration of the results as to speculation in Theorem 3.3 that if we are to require that expectations be fulfilled in equilibrium and that commodity prices be stationary, then it must be the discounted prices which are involved. If current prices were correctly expected to be stationary, total excess demands could not be zero. In fact, in discussing price adjustment in the next section, we shall make the rate of change of discounted prices a sign-preserving function of excess demand, and it will be convergence of discounted prices which is proved as part of the stability result. In equilibrium, current commodity prices are correctly expected to grow at the instantaneous rate given by the profile of bond prices.

Another way of putting it is as follows. We have defined future commodities in terms of their dates rather than in terms of the time period until those dates occur (December wheat, rather than six-month wheat, for example). Under appropriate stationarity conditions (not assumed in this model, but not ruled out either), one might expect the current equilibrium prices of three-month commodities, say, to be constant. This would require the current prices of commodities with a given date to change, however, and it is hardly surprising that the equilibrium rate of change should correspond to the interest rate.

Similarly, in equilibrium it is discounted profits which converge. Current profits grow at the interest rate. Were this not so, firms would either expand or go out of business since the interest rate reflects the opportunity cost of money to all agents.

Discounting also applies to share prices when future dividends are taken into account. The relevant theorem here is Theorem 4.4. I shall return to the subject of discounted versus current prices in considering the price-adjustment equations of the next section. (In the ensuing discussion, I speak of price expectations without always repeating that share price expectations are adjusted for dividends.)

Now, the expectation that current prices will change at the instantaneous interest rate is only required by (7.4) and (7.5) to be held by those agents who are planning corresponding transactions.⁴ What is the

4. It is well to avoid two possible confusions here. In the later discussion, I shall sometimes be comparing two programs, say r and r' , in order to show contradictions emerging from the assumption that r is optimal and r' has certain properties. In such a case, if r' turns out to be optimal, then equilibrium will require constant discounted price expectations for every commodity (or share) and transaction date for which r' has a non-zero transaction even if r has a corresponding zero transaction. In other words, what is required is that such expectations hold where $r(., ., t)$ in the definition is the optimal transactions program, not that it hold for a particular $r(., ., t)$.

Secondly, I shall usually continue to speak as though the optimal program is unique for each agent (possibly because of transaction costs). If there is a set of equally optimal transactions programs at equilibrium, then (7.4) and (7.5) would have to require stationary expectations on discounted price for any agent with some optimal program in which the corresponding transaction was not zero.

meaning of this?

To begin with, neglect smooth transaction costs. It follows from the theorems on speculation (Theorems 3.3 and 4.4) that any agent who does not hold the expectations described in (7.4) and (7.5) must be engaging in transactions in the corresponding commodities or shares until stopped by a constraint. That constraint can be a short constraint, the money constraint (I shall have more to say on this below), or a transaction constraint, if there are any such. In any case, in view of (7.4) and (7.5), such an agent must actually be at a corner solution. So long as his price and dividend expectations are such as to leave him at that corner, the exact expectations he holds will not affect his actions. We cannot expect to prove convergence of such essentially irrelevant expectations, and it would be unreasonable to require specific expectations in equilibrium. The case is rather similar to that of the desire to dispose of free goods (treated below) and, indeed, (7.4) and (7.5) state complementary slackness conditions.⁵

5. In the case of commodities, at least, an agent for whom the short constraint is binding (assuming short sales are allowed) must plan to clear his short position at a later date. It would then follow from (7.4) that his expectations cannot be such as to make the short sales optimal in the first place. This is not true for shares, however, and would not be true for commodities if short sales were illegal; further, a similar statement definitely does not hold as to purchases stopped by the money constraint (but see below). There is no reason why such constraints cannot be binding in equilibrium; corner solutions do not arise only for speculative reasons.

A similar discussion applies to (7.6). If a household's expectations as to the share price and dividends of a firm are not such as to induce him to hold or sell short the shares of that firm, then it is irrelevant what those expectations are and there is no reason to require them to be correct.

There remains one case to consider in which expectations can be for change in discounted commodity prices (or for a parallel change in share prices) and a corner solution still not obtain. That is the case of smooth transaction costs. We saw above that with smooth transaction costs not every expected change in discounted prices leads to action; rather the expected change has to be big enough to overcome the transaction costs involved. (In essence, virtual prices as in (5.2) and (5.3) have to be expected to change.) If prices are not expected to change by enough to warrant action, however, then it is irrelevant by exactly how much they are expected to change. Thus, within the bounds of inertia set by transaction costs, we do not require that price expectations be stationary in equilibrium. Note, however, that, transaction costs or not, we do require stationary expectations if such inertia is overcome.

I shall assume that every agent expects to transact in bonds of every date, even if only because he expects to transfer money forward at the instantaneous interest rate when that date is reached. This means that, in equilibrium, bond market prices are all in equilibrium in the usual sense that a two-year bond bears the same interest rate as the corresponding combination of one-year bonds. It also raises the question of whether the money constraints are binding in equilibrium.

This is a somewhat delicate question which is important for two reasons. The second of these has to do with the convergence proof below and I shall return to it. The first reason, however, has to do with the

sense in which an equilibrium as defined above can properly be called a Walrasian equilibrium even when transaction constraints are not binding. Suppose that a particular agent finds in equilibrium that the money constraint (3.8) or (4.9a) is binding at a finite v . In that case, credit markets are sufficiently imperfect that, if the agent is a household, his budget constraint in equilibrium breaks down into a whole series of budget constraints because he cannot borrow against future sales. Similarly, if the agent is a firm, profits have to be reduced because the firm cannot borrow working capital against future profits. The constraints involved here are both transaction constraints and short constraints on bond and futures dealings (recall that money can always be transferred forward).

That such constraints should be binding is entirely appropriate out of equilibrium where there may be speculative opportunities as shown in Theorems 3.3 and 4.4 and where, in particular, the term structure of bond prices may be viewed differently by different agents. In equilibrium, however, it is less clearly appropriate as such reasons disappear. As it happens, however, it is possible to show that, at least at a Walrasian equilibrium, no money constraint for finite v can ever be binding⁶ except

6. Of course, the money constraint for households is always binding at infinity; this is (4.9b) rather than (4.9a).

for one rather pathological exception. This is true whether or not there are smooth transaction costs.

To see this, first ignore transaction costs. In equilibrium, agents will be indifferent as to the timing of their transactions since there

are no speculative profits to be made. By moving all sales to the present and postponing all purchases to the last possible date, agents can ensure the same utility or profit with the same total cost of purchases out to infinity. Hence, with the exception about to be discussed, the money constraints cannot be binding at finite v . No credit markets need be involved.

Another way of seeing this is as follows. Suppose that the money constraint were binding for some finite v . From the definition of equilibrium and the speculation theorems (Theorems 3.3 and 4.4), the short constraint must also be binding at v for every commodity or share in which the agent wishes to transact at v . But then there is nothing that can be usefully bought with more money at v . (The binding short constraints here correspond to the rearrangement of the timing of purchases in the earlier explanation.) This rather suggests that the money constraint cannot be binding at v .

That suggestion is in fact a proof for firms. Households, however, have another use for money besides purchases, namely, money consumption. It is easy to see that this presents no problem where what is involved is merely the question of whether an addition to the stock of money at v would add to utility through being consumed. A money stock which is only to be consumed will be allocated to money consumption over time so that the marginal utility of money consumption is a constant. But Theorem 4.1 and (4.15) then show that, from the date at which such money begins to be consumed, the money constraint is not binding for finite v (although of course it is so at infinity).

The one catch has to do with another difference between households and firms, the fact that households are allowed to own shares. Suppose that there exists a firm the present value of whose dividend stream is

greater than its share price. In equilibrium, every household owning shares in such a firm will realize this, but there may nevertheless be no excess demand for such shares to affect the price if all such households find the money constraint binding and all other households have different expectations. In that case, accelerating sales and delaying purchases need not help, because the share-owning households are relying on the future dividend stream to finance future consumption.

Notice how peculiar this is, however. So long as the money constraint is binding, Theorem 4.4 assures us that the dividend-corrected price of any share which the household owns or plans to buy must be expected to rise. But this cannot be true in equilibrium of shares which are to be bought, so it applies only to shares being held. In equilibrium, however, the price cannot actually rise so that one would suppose that the shareholders will ultimately change their expectations. Were that to occur, however, the money constraint would cease to be binding with the share price still below the discounted value of the dividend stream. This is clearly impossible if equilibrium is to be maintained since there will then be excess demand for the shares. (This is a case of equilibrium being upset by changing expectations, discussed more fully -- and essentially ruled out -- below.) The alternative is for expectations to remain in error forever and for the money constraint to remain forever binding. In that case, as for firms, there can be no future purchases of anything (although there can be planned future purchases if the rate of change of the dividend-corrected stock price is expected to become zero later) and the dividend stream will in fact be used directly for consumption if equilibrium is maintained. Plainly this is pathological, especially with reasonable credit markets, and I shall say no more about it, although it is not technically necessary to assume it away.

So far this discussion has ignored transaction costs. The somewhat surprising fact is that smooth transaction costs do not change things here despite the fact that some of the argument involved rearranging the timing of transactions. This can be seen in two ways.

The first way is to observe that, in equilibrium, agents can be thought of as maximizing a concentrated utility (or profit) functional as in Section 5. The first-order conditions then apply to that functional with actual rather than virtual prices (the effects of transaction costs having been absorbed into the functionals rather than stated as correcting the prices). But the argument given above from Theorems 3.3 and 4.4 as to the non-binding nature of the money constraints at v rests only on those first-order conditions and must therefore apply.

The other way to think of it applies directly to the rearrangement of timing argument. There it was pointed out that, in the absence of transaction costs, it would always be possible costlessly to rearrange the timing of purchases if there was something to be gained from doing so. With transaction costs, however, and with constant prices, the transaction-costs disadvantage of such rearrangement has already been absorbed into the concentrated utility function. It will still be true that rearrangements will be made without monetary cost if there is something to be gained by doing so, and these are the only costs which matter because the transaction costs themselves have been absorbed into the evaluation of possible gain from rearrangement.

This is all closely related to the not very surprising observation that, in equilibrium with transaction costs, not only will the expected discounted money prices relevant to planned transactions be constant, but so will the corresponding expected virtual prices. With money prices

constant, purchases (or sales) will be arranged to make the marginal disutility of transactions in a particular commodity or stock a constant.

The case is quite different for non-Walrasian equilibria where transaction constraints can keep agents from rearranging the timing of their purchases to alleviate temporary cash shortages. Moreover, the fact that finite-time money constraints are not binding in (Walrasian) equilibria does not justify assuming that they are not binding out of equilibrium, although the same arguments given above would show that they were not so binding at any point at which the agent in question expected prices to remain stationary. Since a principal feature of the present model is that agents need not have such expectations, this creates a substantial difference between the stability proofs given below and those given in earlier papers which did not have to be concerned with such constraints.⁷

7. Arrow and Hahn [1]; Fisher [11, 13, 14, 16, 17]. I discuss this point at length in a later footnote.

Now, turning to another subject, I have defined equilibrium a little vaguely as a "state of the system and its history" rather than in terms of specific variables, because the complex nature of the model means that a set of values for the variables (prices, stocks, expectations, transaction constraints, dividends) which if it occurred at some particular time, t , would produce an equilibrium, will not generally produce one if it occurs at some other time. This is because the history of the system in terms of production and consumption affects the utility functionals being maximized by households and the storage and production

opportunities open to households and firms. While we shall prove below that all such variables converge and the system approaches equilibrium,⁸

8. Incidentally, this (somewhat awkwardly) shows existence of an equilibrium.

the changing nature of the equilibrium set makes it awkward to speak of equilibria in terms only of the current variables.⁹

9. The device used in Fisher [14, 16] of "an equilibrium for t " and "an asymptotic equilibrium" is correct but not very helpful, since the proofs given in those papers are incorrect in the way in which they deal with the historical dependence being discussed.

This raises a somewhat related point. Will an equilibrium, as defined above, be sustained if it is reached? Since this is the question of whether such a point really is an equilibrium of the dynamic system involved, the answer must depend on the equations of motion of the system which have not yet been given. It will be helpful to look ahead, however, and discuss the matter now.

We shall make more-or-less the usual assumptions about price adjustment, so that discounted prices stop moving when excess demands are zero. This will certainly mean that discounted prices will not change so long as (7.1) - (7.3) are satisfied. In turn, this means that the expectations in (7.4) and (7.5) turn out to be correct. Hence every agent finds his expectations fulfilled as to the possibility of transaction completion and as to the prices at which he plans to make non-zero trans-

actions. We have already assumed in discussing the dividend behavior of firms that this implies that firms do not change their dividend plans, so that (7.6) will go on being satisfied if share-owning households do not change their dividend expectations. Households will then receive from firms what they expect to receive.

The catch, of course, lies in the question of whether and how expectations change once equilibrium has been reached. There are two sorts of expectations to consider: those of agents concerning the prices of commodities in which they do not plan to transact or the share price and dividend behavior of firms in which they do not expect to own shares, and those of agents concerning more immediately relevant prices and dividends.

Obviously, the "irrelevant" expectations are not restricted by the definition of equilibrium and they can certainly change.¹⁰ The question

10. But see the related discussion of Assumption 11.6, below.

is whether they can change so as to become "relevant" and induce action. In the context of No Optimistic Surprise (defined rigorously below), the answer is in the negative so long as "relevant" expectations do not change. For if "irrelevant" expectations were to change so as to make it desirable for the agent in question to take action, then they would change so as to make him better off by revealing to him a new opportunity. Moreover, since expectations are assumed to change continuously, "irrelevant" expectations which change so as to become "relevant" would have to first pass through a point at which the agent was just indifferent as to action and no action. If (7.4) and (7.5) are in force, however, then things

can never get further than such a point, since as soon as optimal action tries to turn non-zero, so to speak, price expectations will be stationary.

Thus the really interesting question is whether "relevant" expectations will be maintained in equilibrium. Those expectations, of course, are being fulfilled in equilibrium and it would seem natural to assume that expectations which are fulfilled are preserved. This is not quite so harmless as it may appear, however. The problem is that agents may form their expectations not merely on the basis of their own experience but also on that of other people. Thus, an agent whose expectations are fulfilled as to his own transactions and the relevant prices and dividends may nevertheless revise his expectations when he sees other agents unable to fulfill their demands.¹¹ Indeed, he may revise them if he sees that

11. This is the sort of consideration involved in Maskin's remark to Hahn (discussed above). See Hahn [19].

other agents are able to fulfill their demands if he did not expect this to happen. Clearly, however, this sort of thing is most plausible out of equilibrium. Accordingly, we can assume that, in equilibrium, "relevant" price and dividend expectations do not change.

One further remark on this point. As it happens, such an assumption is not strictly necessary for the stability result below. If we do not make it, then there is no assurance that an equilibrium once reached will be maintained. Even with the assumption of No Optimistic Surprise, there will be nothing to prevent some agent from changing his "relevant" expectations in equilibrium in such a way as to feel himself worse off. However, that same assumption does imply ultimate convergence of all utilities and profits (given that we shall assume them bounded below)

and the stability proof will show that even though the system may pass through other equilibria and leave them because of pessimistic expectation revision, it will eventually converge to one which will not be so left. (Of course this occurs because we do not allow agents to become infinitely pessimistic, but we shall have to assume that in any case.)

Hence, in a sense, it is not necessary to assume that "relevant" expectations do not change in equilibrium, although the sense in which such an equilibrium is properly so-called is then open to question.

Returning to more traditional matters, it is easy to show:

Corollary 7.1: In equilibrium, if there exists a commodity, i , with date θ , and any v ($\theta \geq v \geq t$) such that $R_i(\theta, v, t) < 0$, then $P_i(\theta, t) = 0$. Similarly, if there exists a firm, f , and a time $v \geq t$ such that $Q_f(v, t) < 0$, then $W_f(t) = 0$.

Proof: This follows from the definition of equilibrium and the generalized version of Walras' Law (Theorem 6.1).

As expected, this result states that goods or shares in excess supply will be free in equilibrium. What is perhaps more interesting is that it also states that non-free goods or shares will never be in excess supply once equilibrium is reached, even though trade in them can still take place with the amount of trade different at different future times.

8. Price Change and Trading Rules

I now consider the equations of motion of the system. The present section considers the way in which prices adjust and some of the rules (the more explicit ones) about trades. The next section considers the question of how expectations change and gives an extended discussion of No Optimistic Surprise. The issues treated in the two sections are not unrelated so that, in particular, trade has to take place according to Hahn Process rules for No Optimistic Surprise to make much sense, even though those rules do not have to be explicitly assumed.

The present section deals with more traditional (and possibly simpler) matters, however. I begin with the question of price change. I shall assume (time derivatives are always with respect to t and denoted by dots):

$$(8.1) \quad \dot{P}_i(\theta, t) = H^i(R_i(\theta, t, t), \theta) \text{ unless } P_i(\theta, t) = 0 \text{ and } R_i(\theta, t, t) < 0$$

in which case $\dot{P}_i(\theta, t) = 0$.¹

1. We know from the work of Henry [22, 23] (see also Champsaur, Drèze, and Henry [7]) that this discontinuity does not create special difficulties.

Here $H^i(., .)$ is continuous in both arguments, sign-preserving in its first argument, and bounded away from zero except as $R_i(\theta, t, t)$ approaches zero. $R_i(\theta, t, t)$ is to be understood as total excess demand post-trade at t .

With one important exception, this is the traditional price-adjustment assumption of the stability literature. I add a few comments, which may serve in part to remind the reader of what has already been said.

First, I have avoided the "dated commodities problem" discussed in the introduction by assuming $H^i(., .)$ continuous in its second argument. This means that if $R_i(\theta, t, t)$ is continuous in θ , then a price profile $P_i(\theta, t)$ which starts out continuous in θ can remain so (and I shall assume does remain so) under changes such as (8.1). This only makes sense, however, if one assumes that there is someone in whose interest it is to keep the price of a given commodity continuous in θ ; hence, in the background, we ought to think of price-setting dealers with each commodity being indexed by the dealer who sells it.

Even though it seems most appropriate to think of individuals as setting prices and as having it in their own self-interest to avoid discontinuities in θ , it must nevertheless be remembered that those same individuals do not act optimally in using (8.1) as a price-adjustment rule. Although (see Fisher [11]), we can make the speed of adjustment depend on the individual needs of the price-setter (his liquidity, for example), the fact remains that he takes no account of the fact that, out of equilibrium, he will not necessarily lose all his customers if he raises his price nor is the amount he can sell independent of the price he sets. It is not generally true that monopolists will move prices in the direction indicated by excess demand; optimal pricing on their part depends on elasticity estimates.

If rules such as (8.1) seem somewhat inappropriate where one ought to think of individuals as setting prices, they make more sense in the traditional setting of an impersonal market cleared by an auctioneer. The classic example of such a market is the securities market (where, it is interesting to note, there is no dated commodities problem) and it seems natural to assume:

$$(8.2) \quad \dot{W}_f(t) = F^f(Q_f(t)) - \dot{\bar{G}}(t) \text{ unless } W_f(t) = 0 \text{ and}$$

$$F^f(Q_f(t)) - \dot{\bar{G}}(t) < 0, \text{ in which case } \dot{W}_f(t) = 0.$$

Here $F^f(t)$ is continuous, sign-preserving, and bounded from zero except as $Q_f(t)$ (the post-trade excess demand for the shares of firm f) goes to zero. Note that share prices are automatically corrected for dividend payments as is done in actual securities markets.

The one important respect in which (8.1) and (8.2) differ from the traditional price-adjustment equations of the stability literature is that they are expressed in terms of discounted, rather than current prices. Consideration of the speculation theorems (Theorems 3.3 and 4.4) and the discussion of equilibrium given in the previous section makes it clear that this is the appropriate assumption. Thus, zero excess demand does not lead to zero current price change; rather it leads to current price rising at the instantaneous discount rate given by the bond market.

This is not an anomalous feature. Prices in this model are in terms of current money. There is an opportunity cost to holding money, however; it is the instantaneous discount rate. Indeed, as we saw in discussing the money constraint (3.8), agents will never hold money but will at least move it forward through time by investing in bonds of limitingly short maturity. In effect, therefore, the price of the numéraire changes through time; requiring discounted prices to be constant at zero excess demand requires prices in terms of the true price of the numéraire commodity to be so constant.

Moreover, price-setting agents (if there are any) will consider discounted prices in this way (putting aside the other issues concerning the optimality of the price-adjustment rules already discussed). What

matters to a firm in considering price, for example, is whether price exceeds or falls below costs. "Costs" in that statement, however, always include "normal profit" -- a return on capital which reflects the opportunity costs of the business. In the present model, that opportunity cost is reflected by the opportunities given by the bond market. If, other things equal, prices are set so that firms just recover costs, then prices must rise at the instantaneous discount rate to afford firms the appropriate return on their working capital. If firms are doing the price setting, they will take this into account.

I now turn to the consideration of how trade takes place. As with dividend behavior, discussed in a previous section, it is not necessary to be very precise here. It suffices that trade take place in accordance with certain rules and smoothly enough to permit the existence and continuity of solutions to the differential equations involved. The precise way does not matter.

Such smoothness is by no means trivial, however. We have already assumed (in the notation, for the most part) that agents always have enough goods on hand so that current consumption, storage, and production plans can be fulfilled. This means that agents must not find that, having purchased potatoes, for example, for delivery in September, that their suppliers are unable to deliver when September comes, at least if the buyers plan to consume potatoes. We have already assumed that suppliers are constrained in their planning by the necessity not to be caught short in this way; this is not enough to ensure that they are not in fact so caught. It is very difficult to see how to ensure that frustrations of trading plans are not such as to lead to this kind of situation, and I shall simply have to assume it. Obviously, that assumption is more palatable in a

world in which plans are necessarily consistent with it than in a world in which they need not be.

Related to this is the question of whether individuals are allowed to run out of money -- the No Bankruptcy assumption which plays a central role in Hahn Process models.² We have already assumed that they do not

2. See Arrow and Hahn [1] and Fisher [11, 13, 14, 15, 16, 17].

plan to do so which makes it easier to assume that they do not. Further, unlike the standard Hahn Process model, the assumptions of No Optimistic Surprise in the next section do not explicitly require No Bankruptcy. Those assumptions are generalizations of the Hahn Process assumptions, however, and will not reduce to the latter without a No Bankruptcy assumption; further, it is hard to make the more general assumptions seem plausible without assuming No Bankruptcy. Finally, we at least need to assume that households have enough money on hand to be able to carry out present plans for money consumption, as with ordinary commodities. No Bankruptcy had better be assumed, therefore, but we may note that its special importance has receded somewhat so that it is now pretty much on a par with the other assumptions as to the satisfaction in fact of short constraints which are planned for in advance.³

3. But see the discussion of the money constraint in equilibrium given in the preceding section.

The one rule concerning trading which does need to be explicit is a fairly standard one. Households have already been assumed in (4.1)

to plan to trade things of equal discounted value at constant discounted prices. Now we assume that trade in fact takes place in this way. I shall refer to this as the "No Swindling" assumption; its formal statement is that, for every household (subscript omitted):

$$(8.3) \quad \int_t^{\infty} P(\theta, t) \bar{r}(\theta, t) d\theta + W(t) \bar{q}(t) + \bar{r}_0(t) = 0.$$

There is no need to make a similar assumption for firms; the parallel statement is the time derivative of the definition of actually achieved profits (3.2).

9. No Optimistic Surprise

We must now give a formal description of the way in which expectations change, of the assumption (or collection of assumptions) which has been referred to as "No Optimistic Surprise." In broad outline, that assumption states that, after some finite time (which we may as well take as time 0), agents who alter their expectations always do so in such a way as to perceive themselves worse off than before.¹ Given this,

1. It may seem appropriate that the view that "from here on it is all downhill" should play a central role in an address by one who finds himself elected president of the Econometric Society. "No news is good news" only applies after that.

expectation formation is otherwise unrestricted except insofar as it must be continuous and Lipschitzian in the state variables of the system; there is no need to be more explicit about this.²

2. Notice, however, that this means that the severe underestimation of a transaction constraint at t does not result in a discontinuous shift in expected transaction constraints for times close to t , although it may result in a rapid one.

As remarked in the introduction, it is not contended that such an assumption is an especially plausible one. What is interesting about No Optimistic Surprise is not that it is sufficient for stability, but that its absence is necessary for instability. If what is happening is the disappearance of old opportunities through arbitrage, then the

system will be stable. It is the appearance of real or imagined new opportunities which keeps things moving. Since it is plain that an agent who has money and who suddenly perceives a new opportunity can always upset an equilibrium, this is about as strong a stability result as one can expect without very explicit assumptions as to expectation formation.

Now, the fact that the arbitraging away of old opportunities leads to stability should make one suspect that stability should follow if there is perfect foresight, and, indeed, perfect foresight turns out to be a special case of No Optimistic Surprise. A word of caution must be said about this, however.

We have already remarked that perfect foresight concerning transaction constraints is not an interesting case since it leads to every point being an equilibrium. (As is true in general, it is mistakes which keep things moving.) Accordingly, perfect foresight, if it is to be interesting must mean perfect foresight about prices (and dividends) only. But if (as we have repeatedly suggested lies in the background) prices are set by individuals following the price-adjustment rules of the preceding section, then those individuals, if correctly foreseeing prices, must also foresee the balance of supply and demand which they face (at least as to sign). Hence price-setting dealers can only have perfect foresight about prices if they also have perfect foresight about the transaction constraints on their respective markets. It is true, of course, that each dealer can be mistaken about the transaction constraints which he faces on other markets, but this is a consequence of not allowing individuals to make price offers when hitting such constraints. Thus perfect foresight concerning prices remains an interesting assumption only because we do not have a fully developed and fully individualistic model of price offers.³

3. I am indebted to Gerald Kraft for pointing out these problems to me.

I now turn to the detailed formulation. As before, I begin with the simplest case, that of firms.

Assumption 9.1F:⁴ For every firm, every t , and every $v > t$,

4. I shall refer to Assumptions 9.1F and 9.1H (given below) as Assumption 9.1.

$$(9.1a) \quad \text{If } \mu_0(v, t) > 0, \text{ then } \int_t^\infty \int_t^v \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta \geq 0$$

and

$$(9.1b) \quad \int_t^\infty \int_t^\infty \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta \geq 0,$$

with the equalities holding in (9.1) if and only if the respective integrands are identically zero.

In other words, wherever the money constraint is binding, and certainly asymptotically, changes in price expectations must be unfavorable in the sense that they make the cost of the planned trading program higher. The exception is where changes in expected prices correspond to zero planned trade. Note that perfect foresight as to prices satisfies this assumption, since in that case price expectations never change.

I have stated this assumption in its simplest form, but for most purposes a somewhat weaker version will do. As (9.1) suggests, what

really matters is what happens where money counts. For most purposes, we could replace (9.1) by the weaker

$$(9.2) \quad \int_t^\infty \int_t^\infty \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta \\ + \int_t^\infty \mu_0(v, t) \left\{ \int_t^\infty \int_t^v \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta \right\} dv \geq 0$$

with equality holding if and only if the first integrand is identically zero. Here the change in costs to each date is weighted by the importance of money at that date. It is easy to see from this (as is essentially also true in (9.1)) that the price changes which matter most are those for v close to t . Indeed, by rearranging terms in (9.2), one can see that the weight given to $\dot{p}(\theta, v, t) r(\theta, v, t)$ is precisely $(1 + J(v, t))$. A similar statement applies to the case of the household below, but will not be given explicitly.⁵

5. There the weights would be $\mu_0(v, t)$ and μ_0 and the weight given to changes at v would be $\beta(v, t)$, as one might expect.

Obviously, this assumption (and its companions below) will lead to a situation where changes in expectations make agents worse off. This is particularly clear for firms where profits are directly involved (although even here proof is required since the firm takes actions in response to such changes). In what sort of world does such an assumption make sense?

In the first place, if all agents behave in the way described, then they will find that it was sensible so to do, for if everyone expects old opportunities to be arbitrated away, then the stability theorem below

shows that this is exactly what will happen. This has a strong rational expectations flavor (and, as already remarked, covers the case of perfect foresight).

Secondly, Assumption 9.1F is effectively a generalization of the Hahn Process assumption in its traditional application to the case in which agents naïvely believe that current prices will persist forever. In the latter case, $p_i(\theta, a, t)$ is independent of a and changes in the direction given by aggregate excess demand, $R_i(\theta, t, t)$. In the presence of what I have elsewhere termed the "Present Action Postulate" (Fisher [14, 15, 16]), however, $r_i(\theta, t, t)$ will have the same sign as $\int_t^\infty r_i(\theta, a, t) da$. If we take Assumption 9.1F to apply to each commodity separately, then $r_i(\theta, t, t)$, if non-zero, must have the same sign as $R_i(\theta, t, t)$, which is the Hahn Process assumption.⁶

6. In this discussion it is important to recall that trade takes place instantaneously and that $r_i(\theta, v, t)$ and $R_i(\theta, v, t)$ are post-trade excess demands.

More generally, Assumption 9.1F only makes sense in a Hahn Process world where markets are orderly and only one side constrained.⁷ To

7. Indeed, were this not the case, we could dispense with the standard sign-preserving nature of the price-adjustment process assumed in Section 8.

see this, consider the following.

Suppose that I attempt to purchase⁸ some commodity and am unable

8. In the ensuing discussion, parallel remarks generally hold for attempted sales. There is no point in burdening the text with this, however.

to do so. I may previously have expected the price of that commodity to rise, in which case I presumably expected some agents to have unsatisfied demand. In this case, I may or may not change my expectations on discovering that I am one of the unlucky ones. In any case, provided that I can take my inability to purchase as a signal that there is overall excess demand, it is plausible that I should conclude that the market is at least as restricted as I already thought so that prices will go up at least as fast as previously expected. This leads to a term in the relevant integrals in which a positive excess demand corresponds to a non-negative change in expected price. Moreover, this sort of current experience will lead me to believe that price changes tend to be unfavorable.

On the other hand, if markets are not orderly, there is no reason for me to take my personal inability to purchase as a signal that there is overall excess demand. Indeed, I may very well look at other information and conclude that price will be lower than I had thought. This will lead to a term of the wrong sign in the relevant integrals.

Now, so far, this discussion has considered only the case in which an attempted transaction is not completed. Even if a transaction is completed, however, expectations about future prices may change. How does Assumption 9.1F relate to this?

Suppose that I succeed in buying some commodity. If that commodity is of current date and I am buying it for use as an immediate input (or

for consumption, in the case of a household below), then there are no later-dated prices for it. Further, even if I purchase the good before its date comes due and plan to hold it without selling it again, changes in its expected price cannot violate Assumption 9.1F because all the relevant transaction plans will be zero. There remains the case in which I purchase the commodity planning to sell it later (or sell it planning to buy it back later). In that case, however, it is plausible in a Hahn Process world for me to take completion of my transaction as a signal that there is excess supply rather than excess demand so that prices will fall. At the least, it is plausible that I should not believe that prices will rise faster than I originally thought. This means that my expected price for the date of my planned sale will not rise, which is quite consistent with Assumption 9.1F. In a non-orderly world I may not reach such conclusions.

Unfortunately, however, Assumption 9.1F requires a good deal more than the Hahn Process assumption that markets are orderly (although one should note that it is changes in current prices which are most important). To see this, let us consider what kind of expectations behavior is ruled out.

Suppose that I attempt to buy some commodity believing that the price will rise. If I am unable to complete that transaction, but can only buy some lesser amount, I may then believe that the price will rise even faster than I had originally thought. In that case, while sticking to my original program will make me worse off insofar as my uncompleted purchase will now cost more, I may still be better off because of the unexpected additional capital gain to be made when I sell the part I did manage to buy. This is ruled out by Assumption 9.1F unless it is offset for the terms corresponding

to other commodities or dates. As the total result of such experiences, I must not find myself unexpectedly better off.⁹

9. Strictly speaking, Assumption 9.1F can be violated in this way even if the original transaction is not consummated at all, since what would then be involved would be the change in value of the original program. However, the fact that the expected transaction was not completed would itself make me worse off so that (as used in the proofs below) Assumption 9.1F need not require more than that the net effect be unfavorable.

Obviously, it is a strong assumption that this sort of thing does not happen. If it does, then the unexpected profits involved together with the change in expected prices create a new opportunity, and new action will be taken. If all that is occurring is that old opportunities are being arbitrated away, then this sort of thing is not happening. Note that Assumption 9.1F permits such new opportunities to arise where the money constraint is not binding.

Turning now to households, the parallel assumption to Assumption 9.1F is:

Assumption 9.1H: For every household, every t , and every $v > t$,

(9.3) If $\mu_0(v, t) > 0$, then

$$\int_t^\infty \int_t^v \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta + \int_t^v \dot{w}(a, t) q(a, t) da - \int_t^v \dot{d}(a, t) k(a, t) da \geq 0$$

and

$$\begin{aligned}
 (9.4) \quad & \int_t^\infty \int_t^\infty \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta + \int_t^\infty \dot{w}(a, t) q(a, t) da \\
 & - \int_t^\infty \dot{d}(a, t) k(a, t) da \geq 0,
 \end{aligned}$$

with the equalities holding in (9.3) and (9.4) if and only if every one of the three integrands in those respective expressions is identically zero.

As with the firm, wherever the money constraint is binding (and certainly asymptotically where it must be binding), changes in the relevant expectations make the household worse off in terms of that constraint unless those changes correspond to zero planned quantities. In the case of the household, the expectations involved are for commodity prices, share prices, and dividends, whereas, for the firm, only commodity price expectations were involved.

I have stated (9.3) and (9.4) in one part rather than in three because the theorems below will follow from such a weaker statement, but it is hard to see how those inequalities could plausibly hold generally if similar inequalities did not apply to each of the three integrals involved. Accordingly, the discussion will be in terms of the three separate integrals, ignoring the possibility that gains in one might be offset by losses in the others, just as we have already ignored the possibility that gains in the expected cost of trades in one commodity might be offset by losses in the expected cost of trades in another.

Since the case of commodity price expectations is exactly the same for households as for firms, there is no point in repeating the discussion. Moreover, the same considerations apply to price expectations and planned trades on the securities market, which are involved in the second integral in (9.3) and (9.4). The last integral in those expressions, however,

involves dividend expectations, and this raises some new matters (except where perfect foresight is involved).

We first consider households with non-negative share-holdings. As is fairly obvious, it will turn out below that the profits expected by firms are non-increasing as a consequence of Assumption 9.1F. It thus follows (see (6.8)), that the total dividend payments which any firm expects to make will also be non-increasing. Suppose, first, that the household accepts the firm's estimate of its total future profits. Then the integral in question will be non-positive unless the household expects that the reduction in total dividends will be accomplished by decreasing some payments and increasing others, with the resulting revisions just happening to coincide with the household's planned holdings of the firm's shares in such a way as to make the household receive more dividends than before. This is obviously very implausible, particularly given (9.1a).

The real issue, then, concerns the possibility that the household and the firm do not expect the same total profits. It is clearly enough if the household is sufficiently close to the firm's expectations; further, it is clear that unrealistic optimism on the part of the household is ruled out. Nevertheless, there is more ruled out as well.

Suppose that, initially, the household has more accurate expectations of future prices than the firm. In that case, it will expect lower profits than the firm does. When the firm's expectations change and it lowers its profit expectations, the household may take that as a sign that the firm's managers are becoming more realistic and that the firm will thus be able to make more profit than it (the household) had expected. In this case, the household's dividend expectations may go up even though the firm's are coming down. It may be questioned, however, whether this

is likely to happen to firms in which the household plans to hold shares. After all, unrealistic management and too high profit expectations are unlikely to recommend themselves as an investment. In any case, this sort of thing is ruled out for such firms by Assumption 9.1H.

A parallel, but somewhat different situation holds for households with short positions in securities. Here a fall in profits and actual dividends for the firm means a gain for the household who is holding a negative dividend stream, as it were. However, the expectation of such a fall is at least part of what led to a short position in the first place so such a gain may not be a surprise. If it is, Assumption 9.1H states that it is counterbalanced by less happy ones.

So far, this discussion, like Assumption 9.1 itself, has been in terms of differential movements. We shall sometimes wish to make a stronger assumption concerning discrete time intervals, however.

Assumption 9.1': There exists a time interval, $\Delta^* > 0$, such that for all t^{10} and all Δ such that $0 < \Delta \leq \Delta^*$,

10. Here, and elsewhere, it is only necessary that the requisite properties hold for t large enough. Since we may imagine that we are examining the system after such a large enough date, there is no harm in taking the properties to hold for all t , and it simplifies the exposition.

$$(9.5)(a) \quad \int_{t+\Delta}^{\infty} \int_{t+\Delta}^v \{p(\theta, a, t+\Delta) - p(\theta, a, t)\} r(\theta, a, t+\Delta) da d\theta \geq 0$$

for all $v > t+\Delta$;

$$(b) \quad \int_t^{\infty} \int_t^v \{P(\theta, a) - p(\theta, a, t)\} \bar{r}(\theta, a) da d\theta \geq 0$$

for all $v, t < v \leq t+\Delta$

for all firms, and

$$\begin{aligned}
 (9.6)(a) \quad & \int_{t+\Delta}^{\infty} \int_{t+\Delta}^v \{p(\theta, a, t+\Delta) - p(\theta, a, t)\} r(\theta, a, t+\Delta) da d\theta \\
 & + \int_{t+\Delta}^v \{w(a, t+\Delta) - w(a, t)\} q(a, t+\Delta) da \\
 & - \int_{t+\Delta}^v \{d(a, t+\Delta) - d(a, t)\} k(a, t+\Delta) da \geq 0 \text{ for all } v > t+\Delta;
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad & \int_t^{\infty} \int_t^v \{P(\theta, a) - p(\theta, a, t)\} \bar{r}(\theta, a) da d\theta \\
 & + \int_t^v \{W(a) - w(a, t)\} \bar{q}(a) da \\
 & - \int_t^v \{\bar{d}(a) - d(a, t)\} \bar{k}(a) da \geq 0
 \end{aligned}$$

for all $v, t < v \leq t+\Delta$,

for all households, with the equalities in (9.5a) and (9.6a) holding if and only if all the integrands respectively involved are identically zero.

Assumption 9.1' is stronger than Assumption 9.1 in three respects. First (the minor one), it assumes the inequalities in (9.5a) and (9.6a) to hold for all v , not merely those for which the money constraint is binding. Second, and more serious, it involves a uniform Δ^* for all times and agents. It is easy to see that Assumption 9.1 (applied to all periods irrespective of whether or not the money constraint is binding) implies that, for every t and v and for every agent, there exists a positive Δ^* such that (9.5a) and (9.6a) hold, but while uniformity of that Δ^* over agents is trivial, uniformity over t and v is not. In essence, where

Assumption 9.1 says that the previously optimal program is not getting cheaper, Assumption 9.1' says that the effect is strong enough to last over a discrete interval which does not tend to zero as t and v tend to infinity. ¹¹

11. Such uniformity issues reappear below.

Finally, (9.5b) and (9.6b) say something further. Whereas the rest of the assumption, like Assumption 9.1 itself, talks in terms of the effects of changes in price expectations on the cost of future plans, these two inequalities refer to the effect of actual prices on the cost of actual purchases (and to the effect of actual dividends on income received) between t and $t+\Delta$. They require that the agent be no better off as regards these magnitudes than he would have been had actual prices been what he expected. Put it another way. An agent (by Assumption 9.1 or the remainder of Assumption 9.1') is not made better off, given his purchase plans, by price or dividend changes. On the other hand, he may not be able to complete his transactions. This too will make him no better off. What is ruled out is the possibility that in the short run, those effects offset one another in retrospect so that the agent finds that it would have cost even more to make just those purchases which he actually made had prices not changed adversely.

Were this not to be ruled out, the possibility would exist that the agent finds that some of the short-run money constraints are not as binding as he had expected. This is contrary to the spirit of No Optimistic Surprise, but it could clearly happen by accident. Ruling it out is not needed for most purposes (indeed, use of the differential version, Assumption 9.1,

shows that the net effect of such accidents cannot be to make the agent better off), but is required in one version of the proof of stability rather than quasi-stability, below.

So far, our discussion of No Optimistic Surprise has been entirely in terms of the money constraint. A similar assumption will be used as regards the transaction constraints (5.6) and (5.8).¹²

12. The remaining important constraints in the model are the short constraints. These are assumed to be fixed by law and not to change over time. If they were only perceived constraints, we should need a No Optimistic Surprise assumption as regards them as well.

Assumption 9.2: There exists a time interval, Δ^* , and a scalar $A > 0$, such that for all positive $\Delta < \Delta^*$, all t , all $v \geq t + \Delta$, and all agents,

$$(9.7) \quad b_i(\theta, v, t + \Delta) - b_i(\theta, v, t) \text{ and } s_i(\theta, v, t + \Delta) - s_i(\theta, v, t)$$

are bounded above;¹³

13. Unless $b_i(\theta, v, t)$ or $s_i(\theta, v, t)$ are infinite, respectively. Similarly with (9.10) below.

$$(9.8) \quad \text{if } r_i(\theta, v, t) > b_i(\theta, v, t) - A, \text{ then } b_i(\theta, v, t + \Delta) \leq b_i(\theta, v, t);$$

$$(9.9) \quad \text{if } r_i(\theta, v, t) < -s_i(\theta, v, t) + A, \text{ then } s_i(\theta, v, t + \Delta) \leq s_i(\theta, v, t).$$

Further, for all households,

$$(9.10) \quad B(v, t + \Delta) - B(v, t) \text{ and } S(v, t + \Delta) - S(v, t) \text{ are bounded above;}$$

$$(9.11) \quad \text{if } q_f(v, t) > B_f(v, t) - A, \text{ then } B_f(v, t + \Delta) \leq B_f(v, t);$$

and

$$(9.12) \quad \text{if } q_f(v, t) < -S_f(v, t) + A, \text{ then } S_f(v, t+\Delta) \leq S_f(v, t).$$

I apologize for the complicated way in which this is stated; it is done so as to make the assumption as weak as possible. The simplest assumption that would do here would be the assumption that the perceived bounds on transactions never become looser. Assumption 9.2 only requires this to hold for a positive discrete time interval and, in addition, only requires it for those bounds which the initial desired transactions come close to hitting. Bounds which are sufficiently ineffective are not so restricted.

What can be said about this assumption? Obviously, it is another facet of No Optimistic Surprise; the loosening of a binding transaction constraint represents a new opportunity. In that sense, one might expect such an assumption to be a necessary one for stability. Aside from this, how plausible is the behavior involved?

In the first place, note that Assumption 9.2 will be satisfied if agents do not alter their views concerning transaction constraints. It will also be satisfied if they revise their views in the following manner. Suppose that I begin by attempting to transact. I will do so within my perceived transaction constraints. If I succeed, I may take that as an indication that those constraints were correct. If I do not, then I will have to take that as an indication that constraints were more severe than I had thought. Now, the constraints that can be revised refer to future trades, but the experience I undergo may very well make me believe that errors tend to be in the direction of overestimating my ability to make transactions.

One cannot push this too far, of course; for one thing, such an argument assumes that I do not learn from the experience of others. Assumption 9.2 does allow me to become more optimistic about transaction constraints as a result of such learning, but only where it does not come very close to mattering.

For some purposes, below, it would be possible to cut through many of the complications involved in Assumption 9.2 or Assumption 9.1' by assuming:

Assumption 9.3: There exists a $\Delta^* > 0$, such that, for every agent, every t , and every positive $\Delta \leq \Delta^*$, the program which is optimal at $t+\Delta$ (including its past history from t to $t+\Delta$) was feasible at t .

If we made this even stronger by taking Δ^* infinite and assuming that any program feasible at $t+\Delta$ was feasible at t , it would certainly state very clearly that no new opportunities ever arise. However, in either version, Assumption 9.3 is stronger than needed.

There is one more matter that needs to be taken up before proceeding. The discussion of this section has been entirely in terms of explicit constraints. But agents are also constrained by the available technology. We have assumed throughout that they know all technology with certainty (as they also know their preferences). It would be entirely possible to accommodate changes in technical knowledge, provided that those changes were to make technical constraints more restrictive. Not surprisingly, ^{and unforeseen} new technical discoveries which expand the opportunity set will tend to keep the system moving.

This is obvious in the case of production and storage technology. What may not appear at first sight, however, is that the technology of

transactions is similarly involved. It is this technology which gives rise to the perceived smooth transaction costs. Whether those costs are accurately perceived or not does not matter. If, other things equal, given transactions are continually perceived to have lower costs associated with them than formerly, stability cannot be guaranteed. Accordingly, we have to assume that, after some finite time, transaction costs are perceived not to change favorably. This is, of course, closely related to the perception of transaction constraints, already discussed. A precise statement would involve parametric changes in the Fréchet derivatives of the utility functional (or the production functional) with respect to transactions (see section 5, above), but this is left to the reader. In later sections I shall proceed, for convenience, as though there are no such changes.

10. Profits, Utilities, and Quasi-Stability

In this section, I shall show that target profits for firms (the profits $\pi(t)$ which the firm expects to make if it can complete all its transactions) and target utilities for households (similarly) are non-increasing and ultimately converge. I assume that both target profits and target utilities are bounded below. In examining the behavior of these magnitudes, and in proceeding onwards, it is useful to define:

Definition 10.1: An agent is in personal equilibrium at t if and only if:

- 1) He completes all of his current transactions at t except for the disposal of free goods (and securities, if he is a household).
- 2) For all θ and v , $\theta \geq v \geq t$, a) $r_i(\theta, v, t) \dot{p}_i(\theta, v, t) = 0$;
 b) if the agent is a household, $q_f(v, t) \dot{w}_f(v, t) = 0$
 and $k_f(v, t) \dot{d}_f(v, t) = 0$.
- 3) For all θ and v , $\theta \geq v \geq t$, a) $\zeta(\theta, v, t) \dot{s}(\theta, v, t) = 0$
 and $\xi(\theta, v, t) \dot{b}(\theta, v, t) = 0$;
 b) if the agent is a household, $\tau(v, t) \dot{S}(v, t) = 0$
 and $\psi(v, t) \dot{B}(v, t) = 0$.

An agent is in continual personal equilibrium at t if he is in personal equilibrium from t onwards.

In other words, an agent is in personal equilibrium if he can complete all his current (non-free-disposal) transactions and his relevant price and transaction constraint expectations are not changing. We prove:

Lemma 10.1: If all agents are in continual personal equilibria at t , then the economy is in equilibrium at t .¹

1. The converse is not true unless we assume that relevant expectations do not change once the economy is in equilibrium.

Proof: Since, from t onward, every agent completes his transactions, total excess demand for all non-free goods must be zero and total excess demand for all free goods non-positive. Then discounted commodity prices do not change and discounted share prices change only to reflect dividend payments. Since relevant expectations do not change and transactions are completed, the dividend payments planned by firms do not change either. Further, since every agent knows correctly current prices and dividends, the fact that relevant price and dividend expectations do not change must mean that agents all expected constant discounted relevant commodity prices and (dividend-corrected) relevant share prices and households correctly expected the same dividend plans as did the relevant firms. Since all relevant price and transaction constraint expectations are unchanging, so are optimal programs.² Since the transactions planned

2. Irrelevant expectations do not affect the solution to the agent's optimization problem until they become relevant. This happens with a tangency, rather than a corner solution at zero, so to speak. But at such points the expectations in question cannot change so as to make the formerly zero behavior non-zero or the now just marginally binding transaction constraint actually binding.

under those programs turn out to be feasible when they become current, the planned transactions must have been feasible all along. Comparison of these remarks with Definition 7.1 proves the lemma.

Theorem 10.1: For every firm, $\dot{\pi}(t) \leq 0$, and $\dot{\pi}(t) = 0$ if and only if the firm is in personal equilibrium.

Proof: Target profits change for several reasons. We begin with profits as expected before trade at t . The firm attempts to trade at t . To the extent that it fails, its expected profits post-trade at t must be lower than its expected pre-trade profits, since we have assumed its optimal program to be unique. Second, starting post-trade at t , the firm takes irreversible actions as to the use of currently dated commodities as inputs and outputs. These actions are optimal in the light of the firm's expectations post-trade at t , but they may turn out to be suboptimal if those expectations change. Accordingly, the target profits which the firm will expect before trade at any $t' > t$ can be no greater than they would be if the firm could return to the situation post-trade at t and remake all its decisions on the interval (t, t') in the light of the information acquired in the meantime. To put it another way, profits before trade at t' are, among other things, a functional of the actions of the firm on that interval. They thus cannot be greater with those actions fixed than they would be with those actions remade in an optimal way.

Denote the profits thus replanned as of t by $\pi^*(t)$. I now show that they are not increasing. By the Envelope Theorem, it suffices to examine the way in which the Lagrangian (3.9) (with the addition of the transaction constraint terms (5.7)) changes because of items other than those with respect to which the firm is optimizing. For replannable profits, these are prices, actual stock of commodities, and actual profits and transaction constraints.³

3. It is a mistake to think that one must also differentiate with respect to the lower limits of the various integrals. Think of the integrals over future plans as under the firm's control in the optimum problem. Of course, direct calculation will show that the resulting derivatives cancel out anyway. (The reader who wishes to check this should be careful to note that some integrands are defined as zero for θ or v less than t and Lagrange multipliers should also be so defined, so that integrals over θ , in particular, can be taken to have lower limit zero rather than t .)

$$\begin{aligned}
 (10.1) \quad \dot{\pi}^*(t) = & - \int_0^\infty \int_t^\infty \dot{p}(\theta, v, t) r(\theta, v, t) dv d\theta \\
 & + \int_t^\infty \mu_0(v, t) \left\{ \int_0^\infty \int_t^v \dot{p}(\theta, a, t) r(\theta, a, t) da d\theta \right\} dv \\
 & + \{1 + J(t, t)\} \dot{\pi}(t) + \int_t^\infty \left\{ \lambda(\theta, t) + \int_t^\theta \mu(\theta, v, t) dv \right\} \dot{x}(\theta, t) d\theta \\
 & + \int_t^\infty \int_t^\theta \{ \zeta(\theta, v, t) \dot{s}(\theta, v, t) + \xi(\theta, v, t) \dot{b}(\theta, v, t) \} dv d\theta.
 \end{aligned}$$

where use has been made of (3.13). Using (3.10), (3.2), and (3.7), the terms in $\dot{\pi}(t)$ and $\dot{x}(\theta, t)$ can readily be seen to cancel out, reflecting the fact that when goods are actually bought and sold there are offsetting effects in the firm's accounts involving cash on the one hand and the value of inventories on the other. The remaining terms in (10.1) are all non-positive, however, because of No Optimistic Surprise (Assumptions 9.1F and 9.2).

Since replannable profits are non-increasing, it follows from the remarks already given that actual target profits must also be non-increasing. Further, such target profits will be stationary if and only if replannable profits are stationary, the firm is able to complete its transactions at t , and the irreversible production decisions taken immediately after t are in fact optimal. By No Optimistic Surprise, however, replannable profits can only be stationary if relevant expectations are unchanging. Further, if those expectations are unchanging, then the production decisions taken with them given are in fact optimal. It follows from all this that target profits will be stationary if and only if the firm is in personal equilibrium.

Similarly, we prove:

Theorem 10.2: For every household, $\dot{U}(t) \leq 0$ and $\dot{U}(t) = 0$ if and only if the household is in personal equilibrium.

Proof: The structure of the proof is the same as for firms and will not be repeated in detail. Note that the irreversible decisions which the household takes are those as to consumption and storage. It is only necessary to examine in detail the behavior of replannable utility, $U^*(t)$ (defined analogously to replannable profits), by applying the Envelope Theorem to the Lagrangian (4.10) with the added transaction constraint terms (5.7) and (5.9). The items outside the household's control are prices, transaction constraints, actual holdings of commodities, money, and shares, and dividends.

$$(10.2) \quad \dot{U}^*(t) = - \int_t^{\infty} \beta(a, t) \left\{ \int_0^{\infty} \dot{p}(\theta, a, t) r(\theta, a, t) d\theta + \dot{w}(a, t) q(a, t) \right\} da$$

$$\begin{aligned}
& + \int_t^\infty \mu_0(v, t) \int_t^v \dot{d}(a, t) k(a, t) da dv + \mu_0 \int_t^\infty \dot{d}(a, t) k(a, t) da \\
& + \int_t^\infty \left\{ \lambda(\theta, t) + \int_t^\theta \mu(\theta, v, t) dv \right\} \dot{x}(\theta, t) d\theta \\
& + \beta(t, t) \{ \dot{\bar{m}}(t) - \bar{d}(t) \bar{k}(t) \} + \int_t^\infty \eta(a, t) \dot{\bar{k}}(t) da \\
& + \int_t^\infty \int_t^\theta \{ \zeta(\theta, v, t) \dot{s}(\theta, v, t) + \xi(\theta, v, t) \dot{b}(\theta, v, t) \} dv d\theta \\
& + \int_t^\infty \{ \tau(v, t) \dot{s}(v, t) + \psi(v, t) \dot{B}(v, t) \} dv.
\end{aligned}$$

Here, use has been made of (4.15) and I have changed the notation as to some of the running variables over which integration takes place, for later convenience. Note also that the term in actual dividends, $\bar{d}(t) \bar{k}(t)$, appears as a result of differentiation with respect to the lower limit of integration of the dividend integrals in (4.10). The household cannot control these integrals fully in its optimization problem (as it can integrals involving planned purchases) because they involve the actual holding of securities at t , a stock left over from the past rather than a dated variable.

Now, the term in $\dot{\bar{m}}(t)$ must be understood to involve only those changes in the household's money stock outside the household's control. In particular, current money consumption, $\bar{c}_0(t)$ is within the household's control, as we have assumed it equal to planned consumption. With this understanding, using (4.3), (4.7), (4.8), (4.14), and (4.16), the terms in $\dot{\bar{m}}(t)$, $\dot{\bar{x}}(t)$, and $\dot{\bar{k}}(t)$ can readily be seen to reduce to terms which add to zero by the No Swindling Assumption (8.3) plus a term $\beta(t, t) \bar{d}(t) \bar{k}(t)$ which cancels the similar (but negative) term already present.

The terms involving changes in the perceived transaction constraints are obviously non-positive, by No optimistic Surprise (Assumption 9.2).

There remain the terms in price and dividend changes. Using (4.15) and changing the order of integration in the integrals involving dividend changes, those terms can be written as:

$$(10.3) \quad - \int_t^\infty \beta(a, t) \left\{ \int_0^\infty \dot{p}(\theta, a, t) r(\theta, a, t) d\theta \right. \\ \left. + \dot{w}(a, t) q(a, t) - \dot{d}(a, t) k(a, t) \right\} da$$

which is nonpositive by (4.15) and No Optimistic Surprise (Assumption 9.1H).

The same considerations as in the proof of Theorem 10.1 now show that target utility is non-increasing and strictly decreasing except when the household is in personal equilibrium.

It may be remarked that, unlike the case in Fisher [13, 14, 16, 17], the personal equilibrium for the household involved here need not involve personal equilibria for each of the firms in which it owns shares. This is because the household's expectations as to future dividends need not reflect the actual profits of the firms. This same phenomenon prevents us from simply using the sum of target utilities as a Lyapounov function (this is aside from the problem as to expectations to be discussed in a moment), because that sum can stop moving while firms are not in personal equilibria. Rather (and somewhat inelegantly) we shall use essentially the sum of target utilities and target profits. The apparent discrepancy in units is in fact no worse than that involved in summing utilities over households and, in any case, no economic meaning attaches to the construct.

In fact, not even this sum will quite do as a Lyapounov function. This is because Theorems 10.1 and 10.2 show profits and utilities, respec-

tively, declining when the agent is not in personal equilibrium, whereas Lemma 10.1 only shows the economy to be in equilibrium when all agents are in continual personal equilibria. This corresponds to the possibility that all current demands are feasible and all expectations momentarily unchanging while agents hold mistaken expectations about events some distance in the future.

To take care of this problem, consider the following construct. (I give the discussion explicitly for households in terms of utility; the same construction and discussion applies to firms with profits substituted throughout.) If we assume (as we do) that things are sufficiently smooth that the solution to the differential equations of the system is continuous in its initial conditions, then the position of the state variables of the system after any given time interval, Δ , will also be so continuous. Consider, starting at t , the utility which, when time $t+\Delta$ arrives, the household will then expect to obtain (this is a value of the household's utility functional, U , not an instantaneous utility). Call that value $U(t+\Delta)$. It is continuous in the state variables of the system evaluated at t . Now define:

$$(10.4) \quad V(t) \equiv (1/\rho) \int_0^{\infty} e^{-\rho\Delta} U(t+\Delta) d\Delta,$$

so that $V(t)$ is an exponentially weighted average of the utilities which the household will later come to expect.⁴ It follows immediately from

4. Any positively weighted average (with weights independent of t) will do; it seems neatest to use the discount rate which characterizes the norm in commodity space (see Section 2).

Theorems 10.1 for firms (using profits instead of utilities in (10.4)) and 10.2 for households that:

Lemma 10.2: $\dot{V}(t) \leq 0$ and $\dot{V}(t) = 0$ if and only if the agent is in continual personal equilibrium at t .

Since we are assuming utilities and profits bounded below, this implies the convergence of each $V(t)$. Using the sum of the $V(t)$ over agents as a Lyapounov function, it also follows immediately from Lemmas 10.1 and 10.2 that:

Theorem 10.3: The system is quasi-stable; that is, every limit point of the time path of the system is an equilibrium.

Unfortunately, such quasi-stability is not a very powerful result in the present context, because there is so far no assurance that the time path of the system has any limit points. In fact, such limit points do exist under appropriate assumptions, but the proof that they do is essentially the proof of global stability and makes no use of Theorem 10.3 (although it does rest heavily on the other results of this section).⁵

5. Clearly, the problem is one of compactness. It should be remarked here that the proofs in Fisher [14, 16] are in error on this point (although the theorems themselves are essentially correct). In both papers, compactness of the set of prices is established (in one case because the price space is finite dimensional, in the other because of the use of the weak* topology, together with closure and boundedness in both cases), but the argument that each limit point of prices corresponds to a limit point in consumption and production plans breaks down. This is because the

respective dependence of such plans on the past history of consumption and production means that one cannot simply let prices tend to a limit point without changing that history as the limit is approached. If such history also approached a limit point, there would be no problem, but that is what is to be established.

11. Boundedness and Other Limiting Assumptions

If stability is to hold, then the system must remain bounded. At the level of generality at which we have been working, however, there is nothing so far to ensure that this is so. Essentially, I am going to assume the necessary boundedness (which is boundedness in norm) after a discussion of why it seems plausible to do so.

I begin with prices. Here, it will be recalled, the norm in question will be bounded if the supremum of the undiscounted prices is. Why should this be so?

In earlier work (see Fisher [11, 13, 14, 15]) essentially two ways of establishing boundedness of prices have been suggested. The first way is to assume that if the relative price of money goes to zero there will be excess demand for money. If the price-adjustment processes are specialized in form (they have to lie below some ray in the appropriate space), Walras' Law can be used to show that the sum of squares of the prices remains bounded.

This is not a useful way to proceed in the present analysis. Apart from technical details, the conclusion that could be reached is not the required one. That conclusion would bound the integral of the squared discounted prices whereas what is required is a bound on the supremum of the undiscounted ones which does not follow.

The other existing way of going about it is rather more promising, although still not entirely satisfactory. This is to assume, since we are dealing with relative prices, that if some set of prices gets relatively high enough the commodity (or share) with the highest price has non-positive excess demand, having been "priced out of the market," so to speak. Unfortunately, in the version in which this is most plausible (given in Fisher [13], for example) in which "relatively high enough"

means high relative to all other prices, the proof of boundedness requires a finite number of prices, which we do not have here. Further, even if the assumption were strengthened to mean high relative to the price of money, the conclusion would still only bound the supremum of the discounted prices, since these are what respond to excess demands.

On the other hand, the additional strength of assuming the supremum of undiscounted prices bounded may not be terribly great if we examine the way in which it could fail with the supremum of discounted prices bounded. Plainly, the price for any fixed date, θ , would have to be bounded, since discounting in this case is always back to a fixed date, 0. Further, any eligible price profile has a finite supremum to the undiscounted prices to ensure that prices lie in the normed dual of the space in which the quantity profiles lie and that discounted values of programs are finite. Hence the only way in which the supremum in question can become unbounded is for the date at which some price gets close to it to move ever farther ahead as time goes on. It may not be too much to ask that this not happen when the undiscounted price for every specific date is bounded and undiscounted prices into the indefinite future are bounded as of any specific date, t .

In any case, I shall assume prices and price expectations bounded in this sense.¹

1. As a technical matter, there is no need to do this, since convergence of prices can be proven given the boundedness of the other profiles, but the plausibility of the latter boundedness depends in part on the boundedness of prices, as we are about to see.

For all the other relevant profiles (input, output, storage, consumption, and purchase), called "quantity profiles" for short, the relevant concept is the boundedness of their discounted value (at unit prices) with the discounting always being back to time t so that what is being measured is the "tail" of the profile yet to come. What might make this plausible?

We have already assumed that the solution to each agent's optimum problem is continuous in the prices. Suppose that we were to strengthen this assumption to one of weak* continuity, which does not seem particularly implausible. Since prices are bounded, we can take prices to lie in a closed set, whereupon they will lie in a weak* compact set. It will then follow that, for any t , the norm of the solution to any facet of an agent's optimum problem will be bounded. In other words, faced with bounded prices, agents will not make unbounded plans.

This is not quite enough, however. The solution to the agent's optimum problem depends not only on prices but also on his own past history (consumption and storage decisions in the case of households, production decisions in the case of firms). Even though his plans at any moment may be bounded, cannot the changes in his experience lead that bound to increase without limit?

It is at least plausible that this should not happen. Any plan feasible at time $t' > t$ is part of a plan feasible at t . Since the optimal plan at t is bounded as prices take on all possible variations, it seems plausible that it should not have an unbounded segment starting from t' , particularly given the reductions in opportunities occurring as a consequence of No Optimistic Surprise. There are two possibilities being ruled out here. The first is that even though the optimal program starting from t

was bounded, the program in question would have been suboptimal and even so unbounded. The second has to do with the fact that although the part of the program starting from t' was bounded when discounting back to t , its discounted value back to t' will be bigger. It is possible that as t' goes to infinity such "currently discounted values" grow without limit.

Nevertheless, boundedness seems plausible (and is certainly essential here) and I shall assume it henceforth.

Assumption 11.1 (Boundedness): All price, dividend, and quantity profiles are bounded.

Unfortunately, boundedness and closure (which it would be innocuous to assume) do not guarantee compactness for the set of commodity profiles (although they would guarantee weak* compactness for prices). Accordingly, it is necessary to make some additional assumptions in order to prove stability. Such assumptions would be unnecessary in the presence of compactness; they essentially amount to assuming uniformity in certain properties which have already been assumed to hold at every finite point.²

2. I am indebted to Abigail S. Fisher for pointing out to me that such assumptions cannot reasonably hold without boundedness in the case of strict quasi-concavity below and thus suggesting the importance of first assuming boundedness.

There are some alternative routes to achieve the same end result and I shall state a number of assumptions, not all of which are needed simultaneously.

I begin with the assumptions of No Optimistic Surprise.³ We saw

3. One version of such assumption (Assumption 9.1') is already a uniformity assumption which would be unnecessary in the presence of compactness.

in Theorems 10.1 and 10.2 that expected profits and expected utilities decline except when the agents involved are at points of personal equilibrium. Now I assume that such decline does not approach zero except near personal equilibrium points.

To do this, for any agent, let N^* be the infimum of distances to points of personal equilibrium. Such distances are to be taken as the maximum of the normed difference in the relevant state variables, the latter being: prices; the agent's price, dividend, and transaction constraint expectations; and the agent's holdings of commodities, money, and shares.⁴ In other words, a small N^* means that, given the agent's

4. The infimum is involved because the sets involved are not known to be compact.

production, storage, and consumption history (whichever are relevant), a small difference in the state variables listed would leave him with current transaction plans which can be fulfilled and with unchanging expectations.

We assume:

Assumption 11.2 (Asymptotic No Optimistic Surprise): For any $\delta > 0$, any agent, and any bounded set of states of the system, there exists an

$\varepsilon > 0$ such that $N^* > \delta$ implies $\dot{\pi}(t) < -\varepsilon$ if the agent is a firm and $\dot{U}(t) < -\varepsilon$ if the agent is a household.

Basically, what this assumes is that the No Optimistic Surprise effects (including the surprise of not being able to complete transactions) do not all go to zero except as a point of personal equilibrium is approached.

The next assumption in this line is both more basic and more complicated to state than that just given. We have assumed that utility functionals are strictly quasi-concave. Now it is necessary to ensure that such strict quasi-concavity does not disappear in the limit. For purposes of this assumption, let the arguments of the utility functional (consumptions and transaction costs) be denoted by z . Let $N(z, z')$ be the norm of $(z - z')$.

Assumption 11.3 (Asymptotic Strict Quasi-concavity): For any household, and any $\delta > 0$, let $\{z_\lambda\}$ and $\{z'_\lambda\}$ be any two bounded sequences such that $\lim_{\lambda \rightarrow \infty} U(z_\lambda) = \lim_{\lambda \rightarrow \infty} U(z'_\lambda)$ and $N(z_\lambda, z'_\lambda) > \delta$ for all λ . For any a , such that $0 < a < 1$, there exists an $\varepsilon > 0$ such that, for λ sufficiently large, $U(az_\lambda + (1-a)z'_\lambda) - U(z_\lambda) > \varepsilon$.

In effect, what this says is that indifference curves do not flatten out along bounded infinite sequences. Another way of looking at it is to observe that since, in this model, norms are always being taken by redefining the time origin to count only the future, the utility functional has to remain strictly quasi-concave as we pay more and more attention to later-dated consumption and transactions.

Now, as with strict quasi-concavity itself, it does not really make sense to assume asymptotic strict quasi-concavity when differences only

involve goods whose marginal utility is zero along some relevant path. It is simplest to ignore this for the present and to take care of it in the next section when discussing the treatment of free goods. (A similar remark applies to Assumption 11.4, below.)

It is possible to use Assumption 11.3 to show that the property of strict quasi-concavity that the optimal choice is expenditure-minimizing is preserved asymptotically, but this will not be needed. The parallel property for profit-maximization will be assumed directly and is somewhat simpler. For a firm, let z denote the choice variables in its optimization program (input profiles, output profiles, and sales profiles). Consider $\pi(z)$ as the target profits which the firm expects to make by choosing z .

Assumption 11.4 (Asymptotic Uniqueness of Profit-Maximizing Program):

For any firm and any $\delta > 0$, let $\{z_\lambda\}$ be a bounded sequence of optimally chosen points. Let $\{z'_\lambda\}$ be any corresponding bounded sequence of feasible points such that $N(z_\lambda, z'_\lambda) > \delta$. There exists an $\varepsilon > 0$ such that $\pi(z_\lambda) - \pi(z'_\lambda) > \varepsilon$, where $\pi(z'_\lambda)$ is evaluated at the price expectations at which z_λ was chosen.

In effect, this prevents the profit functional from flattening out and ensures that the uniqueness of the profit-maximizing choice is preserved asymptotically. This could be accomplished by restrictions on isoquants comparable to those which Assumption 11.3 places on indifference curves, so to speak, but it seems simplest to assume it directly.

Finally, many of the special assumptions of this and other sections can be discarded and the proofs much simplified if we assume:

Assumption 11.5 (Uniform Continuity in Money Stock): On the bounded set in which price and quantity profiles remain, the solution to each agent's optimum problem is uniformly continuous in the agent's actual money stock.

This is stronger than necessary, however.

We shall generally require two more asymptotic assumptions. The first is

Assumption 11.6 (Convergence of Expectations): If $\dot{P}(\cdot, t)$ and $\dot{W}(t)$ converge to zero, then, for any agent for which N^* converges to zero also, the difference between expected and actual prices converges to zero. Further, if the agent is a household and, in addition, for every firm $\dot{g}(\cdot, t)$ converges to zero, then the difference between expected and actual dividends converges to zero.

What this says is the following. Suppose that the rate of change of every price goes to zero, so that prices are becoming easier to predict, as it were. Suppose further that the agent in question is coming closer and closer to fulfilling his planned transactions and having his current expectations met. Then the rate of change of his price expectations must eventually also go to zero and expectations will become more and more stationary, a sensible result since this is true of actual prices and the Hahn Process signals received by the agent, so to speak, are for less and less change. Finally, a similar assumption is made as to dividend expectations.

Note that we need only assume that this occurs when it occurs for all markets simultaneously. Given No Optimistic Surprise, such an assumption could be derived from other considerations for all expectations relating

to transactions which are bounded away from zero, but not for transactions which become small. This is related to, but not identical with the question of whether we can allow irrelevant expectations to change once equilibrium is reached. Assumption 11.6 implies that they do not, but its usefulness is for disequilibrium situations.

In any case, we shall need:

Assumption 11.7 (Uniformity of Excess Demands): For any agent, N^* approaching zero implies post-trade excess demands approaching zero. Further, for firms, N^* approaching zero implies $\dot{g}(\cdot, t)$ approaching zero.

This is obviously true if N^* approaching zero means a particular personal equilibrium is being approached. Since we must prove this, Assumption 11.7 states a stronger requirement. For some methods of proof below, Assumptions 11.6 and 11.7 could be largely dispensed with if we could be sure that prices remained positive.

12. Uniqueness of Equilibrium Prices and the Treatment of Free Goods

One further matter remains before proceeding. It is necessary to assume that the economy does not split at equilibrium into two or more disjoint sets of agents and commodities with nothing to connect the relative prices of commodities in the different sets. Such a split could occur if the optimizing problems of one group of agents had corner solutions with respect to one group of commodities while the optimizing problems of the remaining agents had corner solutions with respect to the remaining commodities. Such decomposability would not affect the stability proof as regards commodity profiles but would do so as regards prices.¹ It is simplest to assume:

1. This was first observed in Fisher [11].

Assumption 12.1: At any equilibrium, there is only one set of prices consistent with the optimal program of every agent.

This assumption can be derived from more specific (and complex) assumptions about which of the various first-order conditions for the various optimization problems hold as an equality,² but the conditions

2. As in Fisher [11, 13, 14, 16, 17], for example.

involved are so complex that it seems easiest to assume it directly. As already remarked, what is being ruled out is the concatenation of various corner solutions.

Now, Assumption 12.1 would be entirely innocuous if the only corner solutions that it ruled out were those from short, money, or non-negativity

constraints. In the presence of transaction constraints, however, it is rather more powerful. To see this, observe that one of the possibilities ruled out by Assumption 12.1 is that, in equilibrium, there exists a commodity such that everybody who holds it wants more of it but believes (correctly) that he cannot get it and hence does not try. Similarly, the assumption rules out the opposite case where everyone who has the commodity would like to sell it, nobody wants to buy it, and the sellers are kept from offering it for sale because they believe it will be useless to do so. Both cases are consistent with orderly markets where only one side is transaction-constrained. Assumption 12.1 requires that, in equilibrium, for every commodity and share, there is at least one agent who is satisfied with the amount he has and not merely constrained to have it. Note that, in an orderly market, this will generally be true except where all agents holding the commodity or share are on the same side of the market. Hence the two relatively extreme cases just described are fairly typical of what is ruled out. They are far from impossible, however.

If Assumption 12.1 fails to hold, then it will still be true that consumption, production, share-holding, and related profiles converge as proved below. The adjustment process will only be quasi-stable as regards prices, however. This should not come as a surprise. It corresponds exactly to the proposition that more than one set of prices will support the same corner solutions. The distance of prices from the equilibrium set will approach zero in any case.

Such price indeterminacy, ruled out by Assumption 12.1, has an analogous quantity indeterminacy in that there is nothing to determine the ultimate excess holdings of goods or shares which are free in equilib-

rium. (Note, however, that an agent's holdings of free goods or shares will be determined if he is only kept from increasing them by transaction constraints.) Since it is inconvenient to have to keep raising this as an exception to the convergence results, we build it into the way convergence is measured.

Definition 12.1: In comparing two points, a difference in commodity or share holdings is called inessential relative to a given price system, if: (a) the commodity or share is free in that price system; and, (b) at the point with the lower amount of that commodity or share the transaction constraint on purchasing more of it is never binding.

Definition 12.2: The essential norm of the difference between two quantity profiles relative to a given price system is the norm of that difference less the norm of any inessential differences relative to that price system.

In other words, excess holdings of free goods don't count. The reason for this is that it really does not make sense to have Assumption 11.3 for households or Assumption 11.4 for firms apply when the differences involved are all inessential. Such differences involve zero marginal utilities for households and zero marginal revenue products for both firms and households (For shares they involve zero-priced shares with no return ever expected.). Accordingly, Assumptions 11.3 and 11.4 should be interpreted with $N(z_\lambda, z'_\lambda)$ an essential norm relative to the price expectations at which z_λ was chosen.

We shall show that quantity profiles converge in essential norm relative to a limiting price system.

13. Proof of Global Stability

We shall prove the following theorem:

Theorem 13.1: The model is globally stable. That is, from any initial position it converges to a competitive equilibrium (not necessarily Walrasian if there are transaction constraints). Convergence of prices is in norm and convergence of all other profiles is in essential norm relative to the limiting prices.

The plan of the proof is as follows. First we observe that convergence of utilities and profits, together with Assumption 11.2 (Asymptotic No Optimistic Surprise), implies that the system must eventually have the property that, at every t , it is close to a personal equilibrium for every agent. The bulk of the proof consists of showing that the various quantity profiles converge, so that the equilibria involved are the same for all agents and are continual. This is where the various alternative assumptions as to uniformity come in.¹ Convergence of prices is then handled

1. Interestingly, one device which does not seem available here is the rather pretty one (which originates in Arrow and Hahn [1] and is developed in Fisher [13]) of using expenditure minimization and profit maximization to force a contradiction, given the closed nature of the economy, unless there is convergence. The reason for this is as follows. The fact that production and consumption are taking place out of equilibrium forces us to compare points which are not themselves equilibria (as in Fisher [14, 16]). But out of equilibrium (and even in non-Walrasian equilibrium) there is no assurance that the various money constraints are not binding at finite v . This means that one cannot be sure that equally

good (or almost equally good) programs cost more (or earn less) at the prices of a given program, since they may be unavailable at those prices because they violate the money constraint in finite rather than (as it were) infinite time. Hence the expenditure-minimizing property (or profit-maximizing property) of optimal programs may not hold. I have been unable to get around this problem even by assuming that such money constraints are not binding close to equilibrium (which might be true with transaction costs and loose enough credit markets) because the requisite definition of "close" appears to require assuming the same uniformity which will prove the theorem directly.

The method described will work if one assumes that there is never any binding money constraint at finite v , but this makes sense only if prices are expected not to change and there are no transaction constraints. This is the case, of course, in my previous paper [16], and the proof given there can be readily adapted. (Such adaptation involves observing that compactness -- erroneously there asserted -- is unnecessary since the proof that all limit points are the same will in fact also show convergence directly. Further, prices will converge in norm rather than merely weak*. Finally, as already mentioned, the discussion of norms should be altered. None of these changes would change the general nature of the results, however.)

with Assumption 12.1.

Lemma 13.1: For any agent, N^* , the infimum of distances to points of personal equilibrium converges to zero.²

2. Note that here, as in the proof of quasi-stability (Theorem 10.3),

there is no ambiguity about free goods since it is the infimum of distances which is involved.

Proof: Since we have assumed profits and utilities bounded below, Theorems 10.1 for firms and 10.2 for households ensure that they converge. Assumption 11.2 ensures that this cannot happen unless N^* goes to zero for every agent.

Thus, for every agent and every $\epsilon > 0$, there exists a time after which there is always some personal equilibrium within ϵ of the actual state for that agent, and, since this is true for all agents, it does not matter whether we measure such distances in terms of individual N^* or in terms of the largest N^* over agents. We now proceed to the hard part of the proof, the demonstration that quantity profiles in fact converge so that such equilibria are the same in all relevant respects.

We begin by considering transaction constraints.

Lemma 13.2:³ Under either Assumption 9.2 or Assumption 9.3, there exists,

3. In the statement of results in this section, I mention specifically only those assumptions which have alternatives. Assumptions generally made are not specifically named.

for any agent, a time interval, $\Delta^* > 0$ and a scalar, a^* , $0 < a^* \leq 1$, such that, for all t , all v and θ , $\theta \geq v \geq t$, all Δ , $0 \leq \Delta < \Delta^*$ and all a , $0 < a \leq a^*$:

$$(13.1) \quad b(\theta, v, t) \geq a r(\theta, v, t+\Delta) + (1-a) r(\theta, v, t) \geq -s(\theta, v, t)$$

and (if the agent is a household)

$$(13.2) \quad B(v, t) \geq a q(v, t+\Delta) + (1-a) q(v, t) \geq -S(v, t).$$

Proof: Under Assumption 9.3, this is immediate with $a = 1$. The proof under Assumption 9.2 is more complicated.⁴ I give it explicitly for

4. Note that the complication is entirely due to the weak version of No Optimistic Surprise given in Assumption 9.2. If transaction constraints never get weaker, the lemma is trivial.

firms, the extension to households being immediate.

Define Δ^* and A as in Assumption 9.2. For every commodity, i , and every pair of dates, θ , and v , define $a_i(\theta, v)$ as follows:

Case I: $-s_i(\theta, v, t) \leq r_i(\theta, v, t+\Delta) \leq b_i(\theta, v, t)$. Then $a_i(\theta, v) \equiv 1$.

Case II: $r_i(\theta, v, t+\Delta) > b_i(\theta, v, t)$. Then, if $v \geq t+\Delta$, surely $b_i(\theta, v, t+\Delta) > b_i(\theta, v, t)$, while if $v < t+\Delta$, $b_i(\theta, v, v) > b_i(\theta, v, t)$ (recalling that $r_i(\theta, v, t+\Delta)$ will be actual transactions at v in this case). In either case, Assumption 9.2 implies that $r_i(\theta, v, t) \leq b_i(\theta, v, t) - A$. Choose $a_i(\theta, v)$ such that $a_i(\theta, v) r_i(\theta, v, t+\Delta) + (1 - a_i(\theta, v)) r_i(\theta, v, t) = b_i(\theta, v, t)$. Since $r_i(\theta, v, t+\Delta)$ is bounded above, $a_i(\theta, v)$ is bounded from zero.

Case III: $r_i(\theta, v, t+\Delta) < -s_i(\theta, v, t)$. This parallels Case II and is left to the reader.

Now choose $a^* = \min_i \inf_{\theta, v} a_i(\theta, v)$. Then (13.1) holds by construction.

Thus for short but nonvanishing time intervals, a non-trivial convex combination of the program chosen at the end of the time interval and that chosen at its beginning would have satisfied the transaction constraints perceived at the beginning of the time interval.

Lemma 13.3: Under either Assumption 9.1' or Assumption 9.3, there exists for any agent a time interval, $\Delta^* > 0$, such that, for all t , all Δ , $0 < \Delta < \Delta^*$, a program consisting of actual purchases, holdings (and dividend receipts for households) for all v in the interval $(t, t+\Delta]$ and plans made as of $t+\Delta$ for all $v > t+\Delta$ would satisfy all the money constraints as of t .

Proof: Under Assumption 9.3 this is guaranteed. Consider Assumption 9.1'. It states that the program described would have cost no more up to any future time period had price (and dividend) expectations remained at the values as of t . But, at the later prices and price expectations, the program certainly satisfies the constraints that money stocks be non-negative; this is so up to $t+\Delta$ by the assumption of No Bankruptcy and holds thereafter because the planned program must satisfy such constraints. Thus it would also have been true at the prices expected as of t .

Now, for any agent, let $z(t)$ denote the profiles of the choice variables in his optimization problem, chosen optimally as of t and with actual values before t . For any $\Delta > 0$, define $\hat{N}(\Delta, t)$ as the essential norm of the difference between $z(t)$ and $z(t+\Delta)$ relative to price expectations at t (discounting is to be taken back to t , of course). It is now easy to prove:

Lemma 13.4: Under either Assumption 9.3 or Assumptions 9.1' and 9.2, for any agent, there exists a $\Delta^* > 0$, such that, for any $\epsilon > 0$ and all Δ , $0 < \Delta < \Delta^*$, and for large enough t , $\hat{N}(\Delta, t) < \epsilon$.

Proof: Observe that the set of programs satisfying all the constraints for the agent's optimization problem is convex. By Lemmas 13.2 and 13.3,

there is thus an $a > 0$ such that $\{a z(t+\Delta) + (1-a) z(t)\}$ is feasible at t . However, profits converge, so that the desired result follows from Assumption 11.4 for firms, since a convex combination of two equally profitable programs makes the same profit as either of them. Further, since utilities converge, the desired result follows for households from Assumption 11.3.

We now show that one can obtain essentially the same result without the use of the really strong version of No Optimistic Surprise applied to prices, Assumption 9.1' (Assumption 9.3 is even worse). This involves using the assumptions as to uniform continuity in money stock (Assumption 11.5) and convergence of expectations (Assumption 11.6). }

Lemma 13.5: The conclusion of Lemma 13.4 also follows from Assumptions 9.2, 11.5, 11.6 and 11.7.

Proof: By Lemma 13.2, there is a non-trivial convex combination of $z(t)$ and $z(t+\Delta)$ which satisfies the transaction constraints at t . Call that program \tilde{z} . It approaches $z(t)$ if and only if $z(t+\Delta)$ does.

Now, by Lemma 13.1, N^* approaches zero for every agent. Hence, by Assumption 11.7, every agent's purchase plans get close to what they would be at a point of personal equilibrium. Evidently, then, total current post-trade excess demands must all approach zero (except for excess supplies of goods whose prices approach zero). Hence the rate of change of actual prices approaches zero also. By assumption 11.6, it follows that the rate of change of all price expectations approaches zero. Further, the same must be true of actual and expected dividends.

Suppose then that \tilde{z} would have violated the money constraints with prices and dividends as expected at t . Let \tilde{m} be the additional money which would have to be given to the agent at t to make \tilde{z} satisfy such constraints. Since $z(t)$ satisfies such constraints, \tilde{m} cannot be greater than the amount necessary to make $z(t+\Delta)$ satisfy them. But the rate of change of price and dividend expectations approaches zero, and $z(t+\Delta)$ is bounded. Hence it must be that \tilde{m} approaches zero.

Now consider the program which would have been optimal at t had the agent had a money stock increased by \tilde{m} . Call that program $z^*(t)$. By uniform continuity in money stock (Assumption 11.5), the fact that \tilde{m} approaches zero means that $z^*(t)$ approaches $z(t)$. But then, for a household, utility at $z^*(t)$ approaches utility at $z(t)$, and, for a firm, the parallel statement is true for profits. Since utilities and profits converge, however, this is also the limit of utility or profit attained with $z(t+\Delta)$, as the case may be. Now suppose that $z(t+\Delta)$ did not converge to $z(t)$ and hence to $z^*(t)$. In the case of profits, \tilde{z} would be feasible were money stock increased by \tilde{m} and would asymptotically earn the same profit as the feasible program $z^*(t)$, contradicting Assumption 11.4. In the case of utility, \tilde{z} would be a feasible program under similar circumstances and asymptotically (by Assumption 11.3) would have a higher utility than the supposedly optimal $z^*(t)$, which is a contradiction.

We can now show that Theorem 13.1 follows either from Assumption 9.3 or from Assumptions 9.2 and 9.1', or from Assumptions 9.2 and 11.5.

(Assumptions 11.6 and 11.7 are needed in any case.)

Proof of Theorem 13.1: Under any of the conditions named, there is a fixed $\Delta > 0$ such that $\hat{N}(\Delta, t)$ approaches zero. However, by Lemma 13.1

and Assumption 11.7, the rate of change of actual prices and hence (by Assumption 11.6) of expected prices is approaching zero. Thus, for large enough t , and any $\delta > 0$, any actual or expected price which is greater than δ at t will still be positive at $t+\Delta$. Suppose for a moment that all prices positive at t were still positive at $t+\Delta$. Then $\hat{N}(2\Delta, t) \leq \hat{N}(\Delta, t) + \hat{N}(\Delta, t+\Delta)$, so that $\hat{N}(2\Delta, t)$ must also approach zero. Moreover, the same argument shows that $\hat{N}(b\Delta, t)$ approaches zero for fixed b if all positive prices at t are still positive at $t + b\Delta$. It follows that $z(t + b\Delta)$ converges in essential norm relative to a price vector identical with that of t save for components corresponding to prices which ultimately approach zero, and hence asymptotically relative to the limit of prices at t , if such a limit exists.

Now, consider $z(t + f\Delta)$ where we will let f tend to infinity. Such a program is identical in the interval $[t, t + b\Delta]$ with $z(t + b\Delta)$. Moreover, the norm of any component of the difference between $z(t + f\Delta)$ and $z(t)$ consists of the discounted value of that difference over that same interval plus the discounted value of the difference after $t + b\Delta$, discounting in both cases being taken back to t . But $z(t + b\Delta)$ approaches $z(t)$, so the first part of the difference goes to zero. Moreover, the second part is discounted for a time interval of $b\Delta$, and it is bounded. Since we can make b as large as we like, evidently $z(t + f\Delta)$ converges to $z(t)$ as both t and f go to infinity, convergence being in essential norm as before.

Thus quantity profiles converge in essential norm relative to any price vector which has zero only those prices which converge to zero. However, since N^* approaches zero, the distance between expected prices

(which converge to actual prices) and some set of common expected prices which would make the limit of the quantity profiles a personal equilibrium for every agent must approach zero. Assumption 12.1 states that such prices are unique, however, and the theorem now follows.

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